Adaptive Fuzzy Backstepping Control of Fractional-Order Nonlinear Systems

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***Abstract*—Backstepping control is effective for integer-order nonlinear systems with triangular structures. Nevertheless, it is hard to be applied to fractional-order nonlinear systems as the fractional-order derivative of a compound function is very complicated. In this paper, we develop an adaptive fuzzy backstepping control method for a class of uncertain fractional-order nonlinear systems with unknown external disturbances. In each step, a complicated unknown nonlinear function produced by differentiating a compound function with a fractional order is approximated by a fuzzy logic system, and a virtual control law is designed based on the fractional Lyapunov stability criterion. At the last step, an adaptive fuzzy controller that ensures convergence of the tracking error is constructed. The effectiveness of the proposed method has been verified by two simulation examples.**

***Index Terms*—Adaptive control, backstepping control, fractional order, fuzzy logic, nonlinear system.**

I. Introduction

In THE last 40 years, fractional calculus has captured more and more attention from engineers and physicists due to its interesting properties and potential applications [1]–[9]. It was proved that the fractional differential equation is an effective technique for modeling the dynamics of many chemical properties and special materials [6]. The fractional-order differential equation can be regarded as a generalized model of the conventional integer-order differential equation. More details about the fractional calculus can be referred to [10]–[13]. In recent years, many interesting results have been reported for stability analysis and control of fractional-order nonlinear systems. For example, a fractional-order Lyapunov direct method was proposed in [14]; a state observer was proposed for fractional-order nonlinear systems in [15]; a finite-time stabilization method of fractional-order switched systems was proposed in [16]; adaptive control methods were introduced to control fractional-order nonlinear systems in [10], [11], and [17]; a fuzzy feedback control method of uncertain fractional-order chaotic systems was proposed in [18]. Without exaggeration, we can say that the stability analysis and control of fractional-order nonlinear systems are becoming hot and promising research topics.

It is well known that backstepping control provides an important approach for integer-order nonlinear systems [19]–[21]. This method establishes a systematic framework for controlling a number of nonlinear systems with triangular structure, where its main idea is that some intermediate variables are recursively seen as pseudo control signals. Some researchers have extended the backstepping technique to control fractional-order nonlinear systems [22], [23]. However, only limited types of fractional-order systems with specific assumptions have been investigated. An adaptive backstepping controller similar to those in [22] and [23] was constructed for a class of fractional-order nonlinear systems in [24]. It is worth noting that the existing backstepping control methods which are valid for integer-order nonlinear systems can not be directly extended to fractional-order nonlinear systems because the fractional-order derivative of a quadratic function is very complicated. In [25], an adaptive backstepping controller was designed for a class of uncertain fractional-order nonlinear systems with triangular structure, where integer-order Lyapunov methods are adopted in the stability analysis based on a certain transformation. Thus, how to design an adaptive backstepping controller for triangular fractional-order nonlinear systems based on fractional-order Lyapunov stability criterion is still an open issue.

Integrator backstepping control needs to cancel out system nonlinearities. However, the precise model of system nonlinearities is difficult to be obtained in practice because system uncertainties such parameter variations, modeling errors, and/or external disturbances always exist in nonlinear systems [26]–[34]. Existing control methods considering system uncertainties usually require some prior knowledge such that practical applications of these methods may be restricted while such prior knowledge is not available. Some pioneering results on fuzzy control are combined a fuzzy adaptive controller with a robust compensator, where the former is applied to

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handle system uncertainties, and the latter is mainly motivated by the ability to deal with ubiquitous modeling errors and external disturbances [17], [26], [35]–[55]. Up to now, there exist a few works on adaptive fuzzy control of fractional-order nonlinear systems based on an integer-order Lyapunov function (see [10], [56], [57]). It is known that quadratic Lyapunov functions are frequently used in the stability analysis of adaptive fuzzy control for integer-order nonlinear systems. However, the fractional derivatives of quadratic Lyapunov functions are complicated infinite series. Thus, how to design an adaptive fuzzy controller for fractional-order nonlinear systems based on a fractional Lyapunov stability criterion is still very challenging.

Motivated by the above discussions, an adaptive fuzzy backstepping control method is proposed for a class of uncertain fractional-order nonlinear systems with lower triangular structures in this paper. In each step, a complicated nonlinear function which is produced by differentiating a compound function with a fractional order is approximated by a fuzzy logic system. The contributions of this paper are summarized as follows.

1) An adaptive fuzzy backstepping recursive algorithm is proposed for uncertain fractional-order nonlinear systems, where the stability is analyzed by using the fractional Lyapunov method.
2) Fractional-order nonlinear functions are approximated by fuzzy logic systems, and fractional-order adaptation laws are designed to update the fuzzy parameters.

The rest of this paper is organized as follows. The description of fuzzy logic systems and some preliminaries are presented in Section II. Section III gives the detailed controller design and the stability analysis. Simulation results are shown in Section IV. Finally, Section V concludes this paper.

II. PRELIMINARIES

A. Description of Fuzzy Logic Systems

A fuzzy logic system includes four parts: 1) the knowledge base; 2) the fuzzifier; 3) the fuzzy inference engine working on fuzzy rules; and 4) the defuzzifier, which can be described as follows [58]:

$$f(x(t)) = \frac{\sum_{j\in J} \theta_j(t)\mu_j(x(t))}{\sum_{j\in J} \mu_j(x(t))} \tag{1}$$

in which $\hat{f}$ (a Lipschitz continuous mapping from a compact subset $\Omega \subseteq \mathcal{R}^n$ to the real set $\mathcal{R}$) is the system output, $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in C^1[\mathcal{I}, \Omega]$ (the set of all continuous mappings from $\mathcal{I} = [0, +\infty) \in \mathcal{R}$ to Ω which have continuous derivatives) is the system input, $J = \prod_{i=1}^n \mathcal{F}_i$, $\mathcal{F}_i$ consists of N_i fuzzy sets ($1 \le i \le n$), μ_j (a mapping from $\mathcal{R}^n$ to the closed unit interval [0, 1]) is the membership function of the rule j ($j \in J$). Rule j $(j = (F_1, F_2, \ldots, F_n))$: if $x(t)$ is) $(F_1(t), F_2(t), \ldots, F_n(t))$, then $\hat{f}(x(t))$ is $B^j(t)$), and θ_j (a mapping from $\mathcal{I}$ to $\mathcal{R}$) is the centroid of the jth consequent set ($j \in J$). Let $\theta(t) = [\theta_1(t), \ldots, \theta_N(t)]^T$ and $\vartheta(x(t)) = [q_1(x(t)), q_2(x(t)), \ldots, q_N(x(t))]^T$, where q_j (called the jth fuzzy basis function, $j \in J$) is a continuous mapping (and thus $\vartheta : \Omega \longrightarrow \mathcal{R}^N$ is continuous) defined by

$$q_j(x(t)) = \frac{\theta_j(t)}{\sum_{s\in J} \mu_s(x(t))}.$$

Then, (1) can be rearranged as follows:

$$\hat{f}(x(t)) = \theta^T(t)\vartheta(x(t)). \tag{2}$$

The following lemma demonstrates that the fuzzy logic system is a universal approximator.

Lemma 1 [58]: Assume $f : \Omega \mapsto \mathcal{R}$ is Lipschitz continuous. For each $x \in C^1[\mathcal{I}, \Omega]$ and each $\varepsilon > 0$, there exists a fuzzy logic system in the form of (2) such that

$$\sup_{t\in\mathcal{I}} \left| f(x(t)) - \theta^T(t)\vartheta(x(t)) \right| \le \varepsilon. \tag{3}$$

B. Preliminaries

The fractional-order integrodifferential operator can be seen as an extended concept of the integer-order integrodifferential operator. The fractional-order integral is defined as follows:

$${}_0\mathcal{I}_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \tag{4}$$

where $\Gamma(\alpha) = \int_0^{+\infty} \tau^{\alpha-1} e^{-\tau} d\tau$ is the Euler's Gamma function. The αth Caputo fractional derivative is expressed by

$${}_0^C\mathcal{D}_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau \tag{5}$$

where $n-1 \le \alpha < n$.

Taking Laplace transform on (5) gives

$$\int_0^\infty e^{-st} {}_0^C\mathcal{D}_t^\alpha f(t) dt = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0) \tag{6}$$

where $F(s)$ is the Laplace transform of $f(t)$. In this paper, only the case $\alpha \in (0, 1)$ is involved.

Definition 1 [1]: The Mittag–Leffler function can be given as follows:

$$E_{\alpha,\gamma}(\zeta) = \sum_{k=0}^\infty \frac{\zeta^k}{\Gamma(\alpha k + \gamma)} \tag{7}$$

where α, γ are positive constants, and ζ is a complex number. Its Laplace transform can be given by

$$\mathcal{L}\left\{t^{\gamma-1} E_{\alpha,\gamma}\left(-at^\alpha\right)\right\} = \frac{s^{\alpha-\gamma}}{s^\alpha + a}. \tag{8}$$

Lemma 2 [1]: For a complex number β and two real numbers α, ν satisfying $0 < \alpha < 1$ and

$$\frac{\pi\alpha}{2} < \nu < \min\{\pi, \pi\alpha\} \tag{9}$$

the following equation holds for all integer $n \ge 1$:

$$E_{\alpha,\beta}(\zeta) = -\sum_{j=1}^n \frac{1}{\Gamma(\beta - \alpha j)\zeta^j} + o\left(\frac{1}{|\zeta|^{n+1}}\right) \tag{10}$$

when $|\zeta| \to \infty$, $\nu \le |\arg(\zeta)| \le \pi$.

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Lemma 3 [1]: Let α satisfy $0 < \alpha < 2$ and β be an arbitrary real number. If there exists a positive constant μ such that $(\pi\alpha/2) < \mu \le \min\{\pi, \pi\alpha\}$, then one has

$$\left|E_{\alpha,\beta}(\zeta)\right| \le \frac{C}{1+|\zeta|} \tag{11}$$

where $C > 0$, $\mu \le |\arg(\zeta)| \le \pi$, and $|\zeta| \ge 0$.

Definition 2 [14]: Let $g : [0, b) \longrightarrow \mathcal{I}$ denote a continuous function. If it strictly increases and satisfies $g(0) = 0$, it is said that it belong to class-K.

The following lemma is important for constructing Lyapunov functions to analyze fractional-order nonlinear systems.

Lemma 4 [14]: Assume that the origin is an equilibrium point of a nonautonomous fractional-order nonlinear system

$${}_0^C\mathcal{D}_t^\alpha x(t) = f(t, x(t)) \tag{12}$$

where $f : \mathcal{I} \times \Omega \longrightarrow \mathcal{R}$ is Lipschitz continuous. If there exists a Lyapunov function $V(t, x(t))$ and class-K functions g_i $(i = 1, 2, 3)$ to satisfy

$$g_1(\|x(t)\|) \le V(t, x(t)) \le g_2(\|x(t)\|) \tag{13}$$

$${}_0^C\mathcal{D}_t^\alpha V(t, x(t)) \le -g_3(\|x(t)\|) \tag{14}$$

then (12) is asymptotically stable [i.e., $\lim_{t\to\infty} x(t) = 0$].

Lemma 5 [3]: If $x(t)$ is a smooth function, then

$$\frac{1}{2}{}_0^C\mathcal{D}_t^\alpha\left(x^T(t)x(t)\right) \le x^T(t){}_0^C\mathcal{D}_t^\alpha x(t) \quad (\forall t \in \mathcal{I}). \tag{15}$$

III. Adaptive Fuzzy Backstepping Controller Design

Consider a class of fractional-order nonlinear systems which can be expressed as follows:

$$\begin{cases} {}_0^C\mathcal{D}_t^\alpha x_1(t) = x_2(t) + f_1\left(\underline{x_1}(t)\right) \\ {}_0^C\mathcal{D}_t^\alpha x_2(t) = x_3(t) + f_2\left(\underline{x_2}(t)\right) \\ \quad\vdots \\ {}_0^C\mathcal{D}_t^\alpha x_{n-1}(t) = x_n(t) + f_{n-1}\left(\underline{x_{n-1}}(t)\right) \\ {}_0^C\mathcal{D}_t^\alpha x_n(t) = f_n(x(t)) + d(t) + gu(t) \\ y(t) = x_1(t) \end{cases} \tag{16}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathcal{R}^n$ is the measurable state variable, $y(t) \in \mathcal{R}$ is the output variable, $\underline{x_i}(t) = [x_1(t), x_2(t), \ldots, x_i(t)]^T \in \mathcal{R}^i$, $f_i(\underline{x_i}(t)) \in \mathcal{R}$ is an unknown smooth nonlinear function, $d(t) \in \mathcal{R}$ is an unknown external disturbance, $u(t) \in \mathcal{R}$ is the control input, $g \in \mathcal{R}$ is an unknown positive constant, and $i = 1, 2, \ldots, n$.

Let $y_d(t)$ be a known smooth reference signal. Our objective is to construct a proper controller $u(t)$ such that the tracking error $e(t) := y(t) - y_d(t)$ eventually converges to an arbitrary small region of zero. Then, we give a recursive backstepping algorithm, which can be separated as the following steps.

Step 1: Let the unknown smooth function $f_1(\underline{x_1}(t))$ be approximated by a fuzzy logic system as follows:

$$\hat{f}_1\left(\underline{x_1}(t), \theta_1(t)\right) = \theta_1^T(t)\vartheta_1\left(\underline{x_1}(t)\right). \tag{17}$$

The ideal parameter θ_1^* is given by

$$\theta_1^* = \arg\min_{\theta_1(t)}\left[\sup_{\underline{x_1}(t)}\left|f_1\left(\underline{x_1}(t)\right) - \hat{f}_1\left(\underline{x_1}(t), \theta_1(t)\right)\right|\right]. \tag{18}$$

It should be pointed out that θ_1^* is only for analysis purpose and it is not needed in the control design. Let

$$\tilde{\theta}_1(t) = \theta_1(t) - \theta_1^* \tag{19}$$

$$\epsilon_1\left(\underline{x_1}(t)\right) = \hat{f}_1\left(\underline{x_1}(t), \theta_1^*\right) - f_1\left(\underline{x_1}(t)\right) \tag{20}$$

be the parameter estimation error and the optimal approximation error, respectively. According to the results in [26] and [37], it is feasible to assume that the optimal approximation error remain bounded. As a result, one obtains

$$\left|\epsilon_1\left(\underline{x_1}(t)\right)\right| \le \bar{\epsilon}_1 \tag{21}$$

where $\bar{\epsilon}_1$ is an known positive constant. Then, one gets

$$\begin{aligned} &\hat{f}_1\left(\underline{x_1}(t), \theta_1(t)\right) - f_1\left(\underline{x_1}(t)\right) \\ &\quad = \hat{f}_1\left(\underline{x_1}(t), \theta_1(t)\right) - \hat{f}_1\left(\underline{x_1}(t), \theta_1^*\right) + \hat{f}_1\left(\underline{x_1}(t), \theta_1^*\right) - f_1\left(\underline{x_1}(t)\right) \\ &\quad = \theta_1^T(t)\vartheta_1\left(\underline{x_1}(t)\right) - \theta_1^*\vartheta_1\left(\underline{x_1}(t)\right) + \epsilon_1\left(\underline{x_1}(t)\right) \\ &\quad = \tilde{\theta}_1^T(t)\vartheta_1\left(\underline{x_1}(t)\right) + \epsilon_1\left(\underline{x_1}(t)\right). \end{aligned} \tag{22}$$

It follows from (16) and (22) that

$$\begin{aligned} {}_0^C\mathcal{D}_t^\alpha e(t) &= {}_0^C\mathcal{D}_t^\alpha x_1(t) - {}_0^C\mathcal{D}_t^\alpha y_d(t) \\ &= x_2(t) + f_1\left(\underline{x_1}(t)\right) - {}_0^C\mathcal{D}_t^\alpha y_d(t) \\ &= x_2(t) + f_1\left(\underline{x_1}(t)\right) - \hat{f}_1\left(\underline{x_1}(t), \theta_1(t)\right) \\ &\quad + \hat{f}_1\left(\underline{x_1}(t), \theta_1(t)\right) - {}_0^C D_t^\alpha y_d(t) \\ &= x_2(t) - \tilde{\theta}_1^T(t)\vartheta_1\left(\underline{x_1}(t)\right) - \epsilon_1\left(\underline{x_1}(t)\right) \\ &\quad + \theta_1^T\vartheta_1\left(\underline{x_1}(t)\right) - {}_0^C\mathcal{D}_t^\alpha y_d(t). \end{aligned} \tag{23}$$

Let a virtual control input $\varpi_1(e(t), \underline{x_1}(t), y_d(t))$ be

$$\begin{aligned} \varpi_1\left(e(t), \underline{x_1}(t), y_d(t)\right) = &-\theta_1^T\vartheta_1\left(\underline{x_1}(t)\right) - k_{11}e(t) \\ &- k_{21}\operatorname{sign}(e(t)) + {}_0^C\mathcal{D}_t^\alpha y_d(t) \end{aligned} \tag{24}$$

where $k_{11} > 0$ and $k_{21} > \bar{\epsilon}_1$ are design parameters.

Denote $\varpi_1(t) = \varpi_1(e(t), \underline{x_1}(t), y_d(t))$. Let

$$e_1(t) = x_2(t) - \varpi_1(t). \tag{25}$$

Substituting (24) and (25) into (23) gives

$$\begin{aligned} {}_0^C\mathcal{D}_t^\alpha e(t) = &-\tilde{\theta}_1^T(t)\vartheta_1\left(\underline{x_1}(t)\right) - \epsilon_1\left(\underline{x_1}(t)\right) \\ &- k_{11}e(t) - k_{21}\operatorname{sign}(e(t)) + e_1(t). \end{aligned} \tag{26}$$

Using (21) and multiplying both sides of (26) by $e(t)$ yields

$$\begin{aligned} e(t){}_0^C\mathcal{D}_t^\alpha e(t) &= -e(t)\tilde{\theta}_1^T(t)\vartheta_1\left(\underline{x_1}(t)\right) - \epsilon_1\left(\underline{x_1}(t)\right)e(t) \\ &\quad - k_{21}|e(t)| + e_1(t)e(t) - k_{11}e^2(t) \\ &\le -k_{11}e^2(t) - e(t)\tilde{\theta}_1^T(t)\vartheta_1\left(\underline{x_1}(t)\right) \\ &\quad + \bar{\epsilon}_1|e(t)| - k_{21}|e(t)| + e_1(t)e(t) \\ &\le -k_{11}e^2(t) - e(t)\tilde{\theta}_1^T(t)\vartheta_1\left(\underline{x_1}(t)\right) + e_1(t)e(t). \end{aligned} \tag{27}$$

Choose a Lyapunov function candidate

$$V_1(t) = \frac{1}{2}e^2(t) + \frac{1}{2\sigma_1}\tilde{\theta}_1^T(t)\tilde{\theta}_1(t) \tag{28}$$

and design an adaptation law as follows:

$$ {}_0^C\mathcal{D}_t^\alpha \theta_1(t) = \sigma_1 e(t)\vartheta_1(\underline{x_1}(t)) - \rho_1\theta_1(t) \tag{29}$$

where σ_1 and ρ_1 are positive design parameters. Noting that the fractional-order derivative of a constant is zero and using (19), one immediately obtains

$$ {}_0^C\mathcal{D}_t^\alpha \tilde{\theta}_1(t) = {}_0^C\mathcal{D}_t^\alpha \theta_1(t). \tag{30}$$

Then, according to Lemma 5 and (30), one has

$$\begin{aligned} {}_0^C\mathcal{D}_t^\alpha V_1(t) &= \frac{1}{2}{}_0^C\mathcal{D}_t^\alpha e^2(t) + \frac{1}{2\sigma_1}{}_0^C\mathcal{D}_t^\alpha \tilde{\theta}_1^T(t)\tilde{\theta}_1(t) \\ &\le e(t){}_0^C\mathcal{D}_t^\alpha e(t) + \frac{1}{\sigma_1}\tilde{\theta}_1^T(t){}_0^C\mathcal{D}_t^\alpha \tilde{\theta}_1(t) \\ &= e(t){}_0^C\mathcal{D}_t^\alpha e(t) + \frac{1}{\sigma_1}\tilde{\theta}_1^T(t){}_0^C\mathcal{D}_t^\alpha \theta_1(t). \end{aligned} \tag{31}$$

Substituting (27) and (29) into (31) yields

$$\begin{aligned} {}_0^C\mathcal{D}_t^\alpha V_1(t) \quad &\le -k_{11}e^2(t) + e_1(t)e(t) - \frac{\rho_1}{\sigma_1}\tilde{\theta}_1^T(t)\theta_1(t) \\ &= -\, k_{11}e^2(t) + e_1(t)e(t) - \frac{\rho_1}{\sigma_1}\tilde{\theta}_1^T(t)\tilde{\theta}_1(t) \\ &\quad - \frac{\rho_1}{\sigma_1}\tilde{\theta}_1^T(t)\theta_1^* \\ &\le -\, k_{11}e^2(t) + e_1(t)e(t) - \frac{\rho_1}{2\sigma_1}\tilde{\theta}_1^T(t)\tilde{\theta}_1(t) \\ &\quad + \frac{\rho_1}{2\sigma_1}\theta_1^{*T}\theta_1^* \\ &\le -\,\kappa_1 V_1(t) + e_1(t)e(t) + H_1 \end{aligned} \tag{32}$$

in which $\kappa_1 = \min\{2k_{11}, \rho_1\}$ and $H_1 = (\rho_1/2\sigma_1)\theta_1^{*T}\theta_1^*$ are two positive constants.

Step 2: From (16) and (25), one has

$$\begin{aligned} {}_0^C\mathcal{D}_t^\alpha e_1(t) &= x_3(t) + f_2(\underline{x_2}(t)) - {}_0^C\mathcal{D}_t^\alpha \varpi_1(t) \\ &= x_3(t) + F_2(\underline{x_2}(t)) \end{aligned} \tag{33}$$

in which $F_2(\underline{x_2}(t)) = f_2(\underline{x_2}(t)) - {}_0^C D_t^\alpha \varpi_1(t)$ is an unknown function. Just like the procedures in step 1, let us approximate $F_2(\underline{x_2}(t))$ by using a fuzzy logic system as follows:

$$\hat{F}_2(\underline{x_2}(t), \theta_2(t)) = \theta_2^T(t)\vartheta_2(\underline{x_2}(t)). \tag{34}$$

Then, (33) can be rewritten as follows:

$$\begin{aligned} {}_0^C\mathcal{D}_t^\alpha e_1(t) &= x_3(t) + F_2(\underline{x_2}(t)) \\ &= x_3(t) + F_2(\underline{x_2}(t)) - \hat{F}_2(\underline{x_2}(t), \theta_2(t)) \\ &\quad + \hat{F}_2(\underline{x_2}(t), \theta_2(t)) \\ &= x_3(t) - \tilde{\theta}_2^T\vartheta_2(\underline{x_2}(t)) - \epsilon_2(\underline{x_2}(t)) + \theta_2^T\vartheta_2(\underline{x_2}(t)). \end{aligned} \tag{35}$$

Let a virtual control input be

$$\varpi_2(t) = -\theta_2^T\vartheta_2(\underline{x_2}(t)) - k_{12}e_1(t) - k_{22}\mathrm{sign}(e_1(t)) - e(t) \tag{36}$$

and a fractional-order adaptation law be

$${}_0^C\mathcal{D}_t^\alpha \theta_2(t) = \sigma_2 e(t)\vartheta_2(\underline{x_2}(t)) - \rho_2\theta_2(t) \tag{37}$$

where $k_{12}, \sigma_2, \rho_2 > 0$ and $k_{22} > \bar{\epsilon}_2$ ($\bar{\epsilon}_2$ is a known positive constant satisfying $\|\epsilon_2(\underline{x_2}(t))\| \le \bar{\epsilon}_2$) are design parameters. Choose a Lyapunov function candidate

$$V_2(t) = V_1(t) + \frac{1}{2}e_1^2(t) + \frac{1}{2\sigma_2}\tilde{\theta}_2^T(t)\tilde{\theta}_2(t). \tag{38}$$

Define $e_2(t) = x_3(t) - \varpi_2(t)$. Then, it follows from (32), (35), (36), and (37) that

$$\begin{aligned} {}_0^C\mathcal{D}_t^\alpha V_2(t) &\le -\kappa_1 V_1(t) + H_1 + e_1(t)e_2(t) + e1(t)\varpi_2(t) \\ &\quad + e_1(t)e(t) - e_1(t)\epsilon_2(\underline{x_2}(t)) + e_1(t)\theta_2^T\vartheta_2(\underline{x_2}(t)) \\ &\quad - e_1(t)\tilde{\theta}_2^T\vartheta_2(\underline{x_2}(t)) + \frac{\tilde{\theta}_1^T(t){}_0^C\mathcal{D}_t^\alpha\tilde{\theta}_1(t)}{\sigma_1} \\ &\le -\kappa_1 V_1(t) + H_1 + e_1(t)e_2(t) - k_{12}e_1^2(t) \\ &\quad - \frac{\rho_2}{\sigma_2}\tilde{\theta}_2^T(t)\theta_2(t) \\ &\le -\kappa_2 V_2(t) + H_2 + e_1(t)e_2(t) \end{aligned} \tag{39}$$

where $\kappa_2 = \min\{\kappa_1, 2k_{12}, \rho_2\}$ and $H_2 = H_1 + (\rho_2/2\sigma_2)\theta_2^{*T}\theta_2^*$ are positive constants.

Step i, $3 \le i \le n-1$: Let

$$e_{i-1}(t) = x_i(t) - \varpi_{i-1}(t). \tag{40}$$

Then one has

$$\begin{aligned} {}_0^C\mathcal{D}_t^\alpha e_{i-1}(t) &= x_i(t) + f_i(\underline{x_{i-1}}(t)) - {}_0^C\mathcal{D}_t^\alpha \varpi_{i-1}(t) \\ &= x_i(t) + F_i(\underline{x_i}(t)) \end{aligned} \tag{41}$$

where $F_i(\underline{x_i}(t)) = f_i(\underline{x_{i-1}}(t)) - {}_0^C D_t^\alpha \varpi_{i-1}(t)$ is an unknown function, and $\varpi_{i-1}(t)$ is a virtual control input. Let

$$\hat{F}_i(\underline{x_i}(t), \theta_i(t)) = \theta_i^T(t)\vartheta_i(\underline{x_i}(t)) \tag{42}$$

$$\begin{aligned} \varpi_i(t) &= -\theta_i^T\vartheta_i(\underline{x_i}(t)) - k_{1i}e_{i-1}(t) \\ &\quad - k_{2i}\mathrm{sign}(e_{i-1}(t)) - e_{i-2}(t) \end{aligned} \tag{43}$$

$${}_0^C\mathcal{D}_t^\alpha \theta_i(t) = \sigma_i e(t)\vartheta_i(\underline{x_i}(t)) - \rho_i\theta_i(t) \tag{44}$$

where $k_{1i}, \sigma_i, \rho_i > 0$ and $k_{2i} > \bar{\epsilon}_i$ [$\bar{\epsilon}_i$ is a known positive constant satisfying $\|\epsilon_i(\underline{x_i}(t))\| \le \bar{\epsilon}_i$] are design parameters. Choose a Lyapunov function candidate

$$V_i(t) = V_{i-1}(t) + \frac{1}{2}e_{i-1}^2(t) + \frac{1}{2\sigma_i}\tilde{\theta}_i^T(t)\tilde{\theta}_i(t). \tag{45}$$

Then, after some straightforward manipulations that are very similar to step 2, one obtains

$${}_0^C\mathcal{D}_t^\alpha V_i(t) \le -\kappa_i V_i(t) + H_i + e_i(t)e_{i-1}(t) \tag{46}$$

where $\kappa_i = \min\{\kappa_1, \kappa_2, \ldots, \kappa_{i-1}, 2k_{1i}, \rho_i\}$ and $H_i = H_{i-1} + (\rho_i/2\sigma_i)\theta_i^{*T}\theta_i^*$ are positive constants.

Step n: Let

$$e_{n-1}(t) = x_n(t) - \varpi_{n-1}(t). \tag{47}$$

Then one has

$$\begin{aligned} \frac{1}{g}{}_0^C\mathcal{D}_t^\alpha e_{n-1}(t) &= \frac{1}{g}f_n(x(t)) - \frac{1}{g}{}_0^C D_t^\alpha \varpi_{n-1}(t) + u(t) + \frac{d(t)}{g} \\ &= u(t) + \frac{d(t)}{g} + F_n(t) - \hat{F}_n(t) + \hat{F}_n(t) \\ &= u(t) + d(t) - \tilde{\theta}_n^T\vartheta_n(x(t)) \\ &\quad - \epsilon_n(x(t)) + \theta_n^T\vartheta_n(x(t)) \end{aligned} \tag{48}$$

where $F_n(t) = (1/g)(f_n(x(t)) - {}_0^C D_t^\alpha \varpi_{n-1}(t))$ is an unknown nonlinear function, $\hat{F}_n(t) = \theta_n^T(t)\vartheta_n(x(t))$ is a fuzzy logic systems used to approximate $F_n(t)$, $\tilde{\theta}_n(t) = \theta_n(t) - \theta_n^*$ is a parameter estimation error, and $\epsilon_n(x(t)) = \hat{f}_1(x(t), \theta_n^*) - f_n(x(t))$ is a

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fuzzy approximation error. To proceed, we need the following assumption.

Assumption 1: The external disturbance $d(t)$ is bounded with an unknown upper bound, i.e., $|d(t)/g| \leq \bar{d}$, for all $t \geq 0$ where $\bar{d} > 0$ is an unknown constant.

Let us construct the controller $u(t)$ as follows:

$$u(t) = -\theta_n^T \vartheta_n(x(t)) - k_{1n} e_{n-1}(t) - e_{n-2}(t) - (k_{2n} + \hat{\bar{d}}(t))\text{sign}(e_{n-1}(t)) \quad (49)$$

where $k_{1n} > 0$ and $k_{2n} > \bar{\epsilon}_n$ are design parameters [ϵ_n is a positive constant satisfying $|\epsilon_n(x(t))| \leq \bar{\epsilon}_n$], and $\hat{\bar{d}}(t)$ is the estimation of the unknown constant $\bar{d}$. To update $\hat{\bar{d}}(t)$ and $\theta_n(t)$, design the following fractional-order adaptation laws:

$$ {}_0^C\mathcal{D}_t^\alpha \theta_n(t) = \sigma_n e(t) \vartheta_n(x(t)) - \rho_n \theta_n(t) \quad (50)$$

$$ {}_0^C\mathcal{D}_t^\alpha \hat{\bar{d}}(t) = \eta_1 |e(t)| - \eta_2 \hat{\bar{d}}(t) \quad (51)$$

where σ_n, ρ_n, η_1, and η_2 are positive design parameters.

Substituting (49) into (48) gives

$$\frac{1}{g} {}_0^C\mathcal{D}_t^\alpha e_{n-1}(t) = \frac{d(t)}{g} - \tilde{\theta}_n^T \vartheta_n(x(t)) - \epsilon_n(x(t)) - k_{1n} e_{n-1}(t) - \left(k_{2n} + \hat{\bar{d}}(t)\right)\text{sign}(e_{n-1}(t)) - e_{n-2}(t). \quad (52)$$

Multiplying both sides of (52) by $e_{n-1}(t)$ and using Assumption 1, one obtains

$$\begin{aligned} &\frac{1}{g} e_{n-1}(t) {}_0^C\mathcal{D}_t^\alpha e_{n-1}(t) \\ &\leq \bar{d}|e_{n-1}(t)| - e_{n-1}(t)\tilde{\theta}_n^T \vartheta_n(x(t)) - e_{n-2}(t) e_{n-1}(t) \\ &\quad - \left(k_{2n} + \hat{\bar{d}}(t)\right)|e_{n-1}(t)| - \bar{\epsilon}_n |e_{n-1}(t)| - k_{1n} e_{n-1}^2(t) \\ &\leq -e_{n-1}(t)\tilde{\theta}_n^T \vartheta_n(x(t)) - e_{n-2}(t) e_{n-1}(t) \\ &\quad - \tilde{\bar{d}}(t)|e_{n-1}(t)| - k_{1n} e_{n-1}^2(t) \end{aligned} \quad (53)$$

where $\tilde{\bar{d}}(t) = \hat{\bar{d}}(t) - \bar{d}$ is the estimation error, and k_{2n} is chosen such that $k_{2n} > \bar{\epsilon}_n$. Let

$$V_n(t) = V_{n-1}(t) + \frac{1}{2g} e_{n-1}^2(t) + \frac{1}{2\sigma_n} \tilde{\theta}_n^T(t) \tilde{\theta}_n(t) + \frac{1}{2\eta_1} \tilde{\bar{d}}^2(t). \quad (54)$$

Then, from Lemma 5, (45), (50), and (51), one has

$$\begin{aligned} {}_0^C\mathcal{D}_t^\alpha V_n(t) &\leq -\kappa_{n-1} V_{n-1}(t) - e_{n-1}(t)\tilde{\theta}_n^T \vartheta_n(x(t)) \\ &\quad + \frac{1}{\sigma_n} \tilde{\theta}_i^T(t) {}_0^C\mathcal{D}_t^\alpha \theta_i(t) + \frac{1}{\eta_1} \tilde{\bar{d}}(t) {}_0^C\mathcal{D}_t^\alpha \hat{\bar{d}}(t) \\ &\quad - \tilde{\bar{d}}(t)|e_{n-1}(t)| - k_{1n} e_{n-1}^2(t) + H_{n-1} \\ &\leq -\kappa_{n-1} V_{n-1}(t) + H_{n-1} - k_{1n} e_{n-1}^2(t) \\ &\quad - \frac{\rho_n}{\sigma_n} \tilde{\theta}_n^T(t) \theta_n(t) - \frac{\eta_2}{\eta_1} \tilde{\bar{d}}(t) \hat{\bar{d}}(t) \\ &\leq -\kappa_{n-1} V_{n-1}(t) + H_{n-1} - k_{1n} e_{n-1}^2(t) \\ &\quad - \frac{\eta_2}{2\eta_1} \tilde{\bar{d}}^2(t) + \frac{\eta_2}{2\eta_1} \bar{d}^2 + \frac{\rho_n}{2\sigma_n} \theta_n^{*T} \theta_n^* \\ &\quad - \frac{\rho_n}{2\sigma_n} \tilde{\theta}_n^T(t) \tilde{\theta}_n(t) \\ &\leq -\kappa_n V_n(t) + H_n \end{aligned} \quad (55)$$

in which $\kappa_n = \min\{\kappa_{n-1}, 2k_{1n}, \rho_n, \eta_2\}$ and $H_n = H_{n-1} + (\eta_2/2\eta_1)\bar{d}^2 + (\rho_n/2\sigma_n)\theta_n^{*T}\theta_n^*$ are two positive constants. The stability result of the closed-loop system is given as follows.

Theorem 1: For the system (16) under Assumption 1, if the control input is designed as (49) with (24), (36) and (43), and the adaptation laws are chosen as (26), (37), (44), (50), and (51), then the tracking error $e(t)$ tends to an arbitrary small region of the origin under a proper choice of design parameters.

Proof: According to (55), one obtains

$$ {}_0^C\mathcal{D}_t^\alpha V_n(t) + m(t) = -\kappa_n V_n(t) + H_n \quad (56)$$

where $m(t) \geq 0$. Taking Laplace transform on (56) gives

$$\begin{aligned} V_n(s) &= \frac{s^{\alpha-1}}{s^\alpha + \kappa_n} V_n(0) + \frac{H_n}{s(s^\alpha + \kappa_n)} - \frac{M(s)}{s^\alpha + \kappa_n} \\ &= \frac{s^{\alpha-1}}{s^\alpha + \kappa_n} V_n(0) + \frac{s^{\alpha-(1+\alpha)} H_n}{(s^\alpha + \kappa_n)} - \frac{M(s)}{s^\alpha + \kappa_n} \end{aligned} \quad (57)$$

where $V_n(s)$ and $M(s)$ are Laplace transforms of $V_n(t)$ and $m(t)$, respectively. Using (8), one can solve (57) as follows:

$$V_n(t) = V_n(0) E_{\alpha,1}\left(-\kappa_n t^\alpha\right) + H_n t^\alpha E_{\alpha,1+\alpha}\left(-\kappa_n t^\alpha\right) - m(t) * t^{-1} E_{\alpha,0}\left(-\kappa t^\alpha\right) \quad (58)$$

where $*$ represents the convolution operator. Noting $m(t)$ and $t^{-1}E_{\alpha,0}(-\kappa_n t^\alpha)$ are both non-negative functions, one has $m(t) * t^{-1}E_{\alpha,0}(-\kappa_n t^\alpha) \geq 0$. Then, one obtains

$$|V_n(t)| \leq |V_n(0)| E_{\alpha,1}\left(-\kappa_n t^\alpha\right) + H_n t^\alpha E_{\alpha,1+\alpha}\left(-\kappa_n t^\alpha\right). \quad (59)$$

Noting that $\arg(-\kappa_n t^\alpha) = -\pi$, $|-\kappa_n t^\alpha| \geq 0$ for all $t \geq 0$ and $\alpha \in (0, 2)$ and Lemma 3, one knows that there exists a positive constant C such that

$$\left|E_{\alpha,1}\left(-\kappa_n t^\alpha\right)\right| \leq \frac{C}{1 + \kappa_n t^\alpha}. \quad (60)$$

It follows from (60) that

$$\lim_{t\to\infty} |V_n(0)| E_{\alpha,1}\left(-\kappa_n t^\alpha\right) = 0. \quad (61)$$

Hence, for every $\varepsilon > 0$, there exists a constant $t_1 > 0$ such that $t > t_1$, which implies

$$|V_n(0)| E_{\alpha,1}\left(-\kappa_n t^\alpha\right) < \frac{\varepsilon}{3}. \quad (62)$$

On the other hand, using Lemma 2, one has

$$E_{\alpha,\alpha+1}(-\kappa_n t^\alpha) = \frac{1}{\Gamma(1)\kappa_n t^\alpha} + o\left(\frac{1}{|\kappa_n t^\alpha|^{1+1}}\right) \quad (63)$$

where the integer n in Lemma 2 is chosen as $n = 1$. From (63), for every $\varepsilon > 0$, there exists a positive constant t_2 such that

$$H_n t^\alpha E_{\alpha,\alpha+1}\left(-\kappa_n t^\alpha\right) \leq \frac{H_n}{\kappa_n} + \frac{\varepsilon}{3} \quad (64)$$

for all $t > t_2$. As the design parameters can be adjusted such that $(H_n/\kappa_n) \leq (\varepsilon/3)$, it follows from (59), (62), and (64) that

$$|V_n(t)| < \varepsilon. \quad (65)$$

From (65) and the definition of $V_n(t)$, one gets that all signals in the closed-loop system keep bounded and the tracking

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error $e(t)$ tends to an arbitrary small region of the origin $[(1/2)e^2(t) \leq \varepsilon$ for all $t > \max\{t_1, t_2\}]$ eventually. END. ■

Remark 1: It should be mentioned that a backstepping control method was also proposed for a class of fractional-order nonlinear systems in [25]. In [25], by using a kind of system transformation, the integer-order Lyapunov method is used in the stability analysis, where the knowledge of the system model should be exactly known. However, in our work, the system model can be fully unknown. In addition, by proposing a kind of fractional-order adaptation laws, a fractional-order stability criterion can be used in the stability analysis.

Remark 2: The proposed method is also valid for the following class of systems after some simple modifications:

$$\begin{cases} {}_0^C\mathcal{D}_t^\alpha x_1(t) & = g_2 x_2(t) + f_1(\underline{x_1}(t)) \\ {}_0^C\mathcal{D}_t^\alpha x_2(t) & = g_3 x_3(t) + f_2(\underline{x_2}(t)) \\ & \vdots \\ {}_0^C\mathcal{D}_t^\alpha x_{n-1}(t) & = g_n x_n(t) + f_{n-1}(\underline{x_{n-1}}(t)) \\ {}_0^C\mathcal{D}_t^\alpha x_n(t) & = f_n(x(t)) + d(t) + u(t) \\ y(t) & = x_1(t) \end{cases}$$

where $g_2, g_3, \ldots, g_n$ are known constants.

Remark 3: It should be pointed out that the result of Theorem 1 can be conveniently used to analyze the stability of fractional-order systems by using a fractional Lyapunov stability criterion. It can be concluded that if there exist two positive constants ς_1 and ς_2 such that ${}_0^C\mathcal{D}_t^\alpha V(t) \leq -\varsigma_1 V(t) + \varsigma_2$, then $z(t)$ is globally bounded and $z(t) \leq (\varsigma_2/\varsigma_1)$ when t is large enough, where $V(t) = (1/2)z^T(t)z(t)$ with $z(t) \in \mathcal{R}^n$ is a Lyapunov function.

IV. Simulation Studies

To show the effectiveness of the proposed method, we use two simulation examples in this section.

A. Example 1

Consider a fractional-order nonlinear system as follows:

$$\begin{cases} {}_0^C\mathcal{D}_t^\alpha x_1(t) = x_2(t) - 0.4x_1^2(t) \\ {}_0^C\mathcal{D}_t^\alpha x_2(t) = -0.1x_2(t) + \frac{x_2(t) - 0.5x_1^2(t)}{1 + x_1^4(t)} + d(t) + u_2(t). \end{cases} \tag{66}$$

Let the initial condition be $x(0) = [-3, 5]^T$, and the fractional order be $\alpha = 0.95$. As illustrated in Fig. 1, when $d(t) \equiv u(t) \equiv 0$, the uncontrolled system (66) is unstable.

The design parameters are chosen as $k_1 = k_2 = 1, k_{11} = k_{12} = 1$, $\sigma_1 = \sigma_2 = 5$, and $\rho_1 = \rho_2 = 0.5$, where small control gain parameters are used to show the effective approximation ability of the employed fuzzy logic systems. Let $y_d(t) = \sin t$ and $d(t) = \sin t + \cos t$. To avoid the chattering phenomenon, $\text{sign}(\cdot)$ is replaced by $\arctan(10\cdot)$ in the proposed controller.

There are two fuzzy systems used in the proposed controller. The first fuzzy system employs $y(t)$ as its input and defines six Gaussian membership functions as $\exp(-(x - c_i)^2/(2\sigma_i^2))$ with $i = 1, 2, \ldots, 6$, uniformly distributed on $[-5, 5]$. The membership functions are shown in Fig. 2. The initial condition is chosen as $\theta_1(0) = [1, 1, 1, 1, 1, 1]^T$. The second fuzzy system uses $x_1(t)$ and $x_2(t)$ as its inputs. For each input, the membership functions are chosen as the same as the first one. The initial condition is chosen as $\theta_2(0) = [1, \ldots, 1]^T \in \mathcal{R}^{36}$.

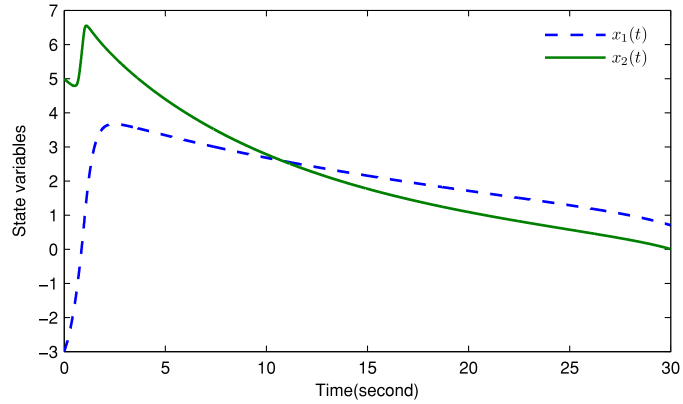

Fig. 1. Time response of state variables of the uncontrolled system (66).

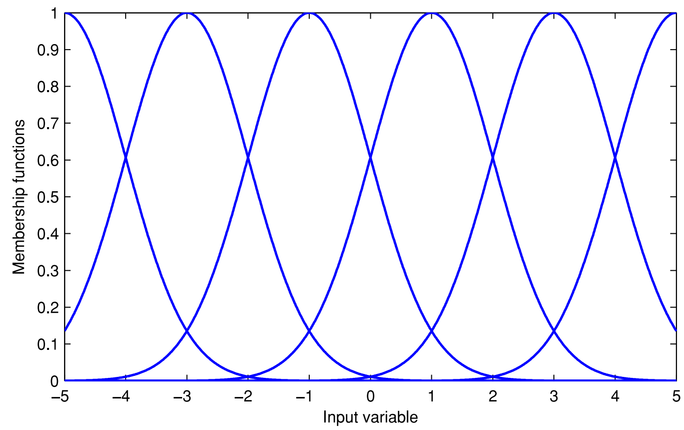

Fig. 2. Membership functions of fuzzy logic systems.

Simulation results are presented in Fig. 3. One sees that the tracking error $e(t)$ has a rapid convergence, and the controller works well even in a noisy environment with a fully unknown system model. However, $e(t)$ cannot stop at the 0 but has tiny fluctuations near 0. The reasons for this result consist in: 1) for a fractional-order system, whenever the system trajectory reaches the equilibrium 0, it cannot stay there thereafter as there are no finite-time stable equilibria in fractional-order nonlinear systems [2] and 2) $\text{sign}(\cdot)$ is replaced by $\arctan(10\cdot)$ so that asymptotical convergence of the tracking error cannot be guaranteed. It is also shown in Fig. 3 that the control input and the parameters of the fuzzy systems remain bounded.

B. Example 2

Consider a fractional-order Chua–Hartley's system [59]

$$\begin{cases} {}_0^C\mathcal{D}_t^\alpha x_1(t) = x_2(t) + \frac{10}{7}\left(x_1(t) - x_1^3(t)\right) \\ {}_0^C\mathcal{D}_t^\alpha x_2(t) = x_3(t) + 10x_1(t) - x_2(t) \\ {}_0^C\mathcal{D}_t^\alpha x_3(t) = -\frac{100}{7}x_2(t) + d(t) + u(t). \end{cases} \tag{67}$$

When $u(t) = d(t) \equiv 0$ and $\alpha = 0.98$, system (67) shows rich dynamical behavior, which is illustrated in Fig. 4.

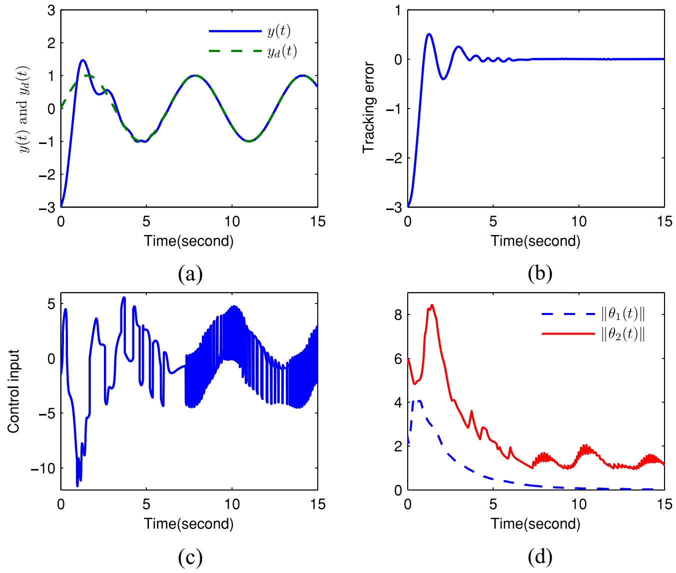

Fig. 3. Simulation results of Example 1. (a) $y(t)$ and $y_d(t)$. (b) Tracking error. (c) Control input. (d) Fuzzy parameters.

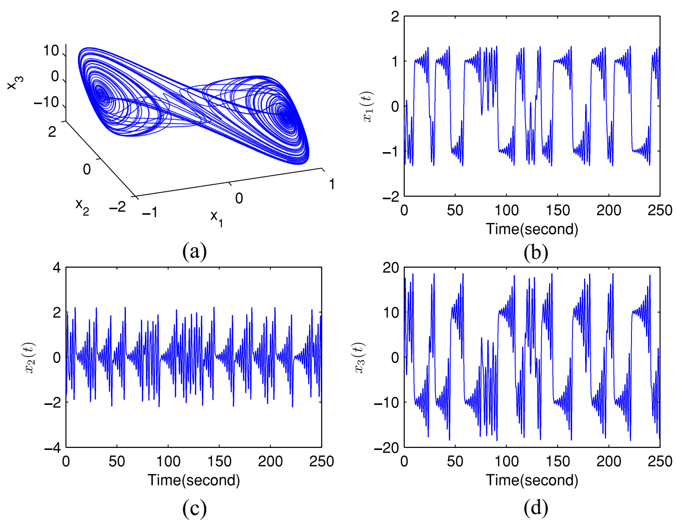

Fig. 4. Dynamical behavior of the uncontrolled fractional-order Chua–Hartley's system (67).

The design parameters are chosen as $k_1 = k_2 = 1.6, k_3 = 3, k_{11} = k_{12} = 1.6, k_{13} = 3, \sigma_1 = \sigma_2 = \sigma_3 = 2$, and $\rho_1 = \rho_2 = \rho_3 = 0.2$. Let $y_d(t) = \cos t$ and $d(t) = 2 \sin t \cos t$. The initial condition is set as $x(0) = [-2, -1, 1]$.

There are three fuzzy systems used in this case. The first fuzzy system employs $y(t)$ as its input, and defines four Gaussian membership functions uniformly distributed on $[-3, 3]$ as in Example 1. The initial condition is chosen as $\theta_1(0) = [1, 1, 1, 1]^T$. The second fuzzy system uses $x_1(t)$ and $x_2(t)$ as its inputs. For each input, the membership functions are chosen as the same as the first one. The initial condition is chosen as $\theta_2(0) = [1, \ldots, 1]^T \in \mathcal{R}^{16}$. The last fuzzy system utilizes $x_1(t)$, $x_2(t)$ and $x_3(t)$ as its inputs. With respect to $x_3(t)$, we define five Gaussian membership functions uniformly distributed on $[-10, 10]$. The initial condition is chosen a $\theta_3(0) = [1, \ldots, 1]^T \in \mathcal{R}^{80}$. Simulation results of this example

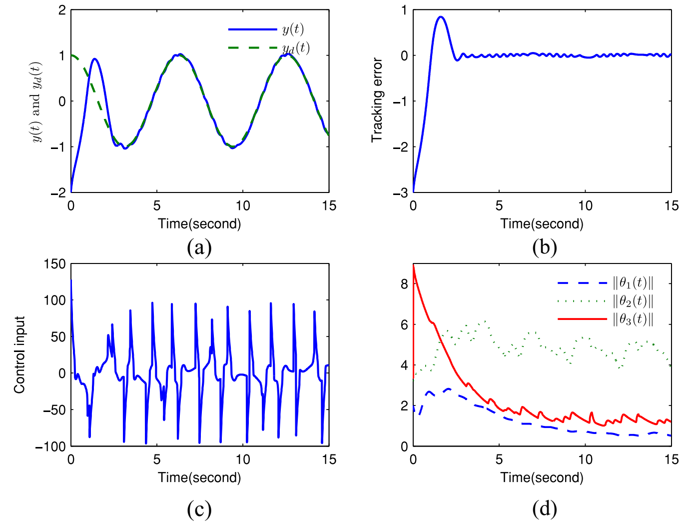

Fig. 5. Simulation results of Example 2. (a) $y(t)$ and $y_d(t)$. (b) Tracking error. (c) Control input. (d) Fuzzy parameters.

are shown in Fig. 5, where qualitative analysis of these results is the same as that of Example 1.

V. Conclusion

In this paper, we have presented an adaptive fuzzy backstepping control method for a class of fractional-order nonlinear systems with triangle structures. In each step, a fuzzy system is used to approximate an unknown nonlinear function. Finally, a robust adaptive fuzzy controller which guarantees convergence of the tracking error is constructed based on a fractional Lyapunov stability criterion. Our result indicates that the backstepping control technique can be extended to fractional-order nonlinear systems based on the fractional Lyapunov stability criterion and fractional-order adaptation laws. How to design an adaptive fuzzy backstepping controller for fractional-order nonlinear systems with input nonlinearities (such as input saturation, dead zone, and backlash-like hysteresis) is one of our future research directions.

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