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Fractional-order Lagrange polynomials: An application for solving delay fractional optimal control problems

Sedigheh Sabermahani[1], Yadollah Ordokhani[1] and Sohrab-Ali Yousefi[2]

Abstract

The main purpose of this work is to provide an efficient method for solving *delay fractional optimal control problems* (DFOCPs). Our method is based on *fractional-order Lagrange polynomials* (FLPs) and the collocation method. The FLPs are used to achieve a new operational matrix of fractional derivative. Also, we present a delay operational matrix of FLPs. These operational matrices are driven without considering the nodes of Lagrange polynomials. The operational matrices and collocation method are applied to a constrained extremum in order to minimize the performance index. Then, the problem reduces to the solution of a system of algebraic equations. Convergence of the algorithm and approximation of FLPs are proposed. Furthermore, the upper bound of the error for the operational matrix of fractional derivatives is obtained. Numerical tests for demonstrating the efficiency and effectiveness of the method are included. Moreover, the method is used for numerical solution of a mathematical model of chemotherapy in breast cancer.

Keywords

Delay fractional optimal control problems, fractional-order Lagrange polynomials, operational matrix, convergence analysis, numerical solution

Introduction

Time-delay systems have been very much considered in the last few decades. Many of these time-delay systems appear in different systems and branches of science such as engineering, chemistry, physics, disease models (Zhang et al., 2014), traffic control (Sipahi and Niculescu, 2009), etc.

The theory of a general class of differential equations with delayed arguments was introduced in 1949 by Myshkis (1949). Furthermore, Krasovskii (1963), Bellman and Cooke (1963), El'sgol'c and Norkin (1971), and Hale (1977) researched in this field.

Analysis and control design are very complicated in the presence of delay, so a time-delay system is very important for many researchers to control, stabilize and optimize them (Lin et al., 2006; Malek-Zavarei and Jamshidi, 1987; Fischer and Nappo, 2008).

On the other hand, in recent decades fractional calculus has attracted many researchers. In the early ages of modern differential calculus, right after the introduction of $\frac{\mathrm{d}}{\mathrm{d}x}$ for the first derivative, in a letter dated 1695, Hopital asked Leibniz the meaning of $\frac{\mathrm{d}^{0.5}}{\mathrm{d}x^{0.5}}$, the derivative of order 0.5. The appearance of 0.5 as a fraction gave the name *fractional calculus* to the study of derivatives, and integrals, of any order, real or complex. There are several different approaches and definitions in fractional calculus for derivatives and integrals of arbitrary order.

Attempting to answer the question of Hopital, Leibniz tried to explain the possibility of the derivative of order 0.5. He also quoted that "this will lead to a paradox with very useful consequences". During the next century the question was raised again by Euler (1738), Laplace (1812), Lacriox (1819), etc. expressing an interest in the calculation of fractional-order derivatives. The first challenge of making a definition for arbitrary order derivatives comes from Fourier in 1822. For the first time, Liouville (1832) presented definitions of fractional operators. Many mathematicians, like Peacock and Kelland (1839), and Gregory (1841) have contributed to this. Riemann dedicated the introduction of fractional derivatives and integrals of Riemann–Liouville to Sonin in 1869, and in some work to Pooseh (2013).

Fractional calculus appears in the modelling of many phenomena. Indeed, a strong motivation to study and research to solve fractional differential equations with time delay comes from the fact that these equations efficiently describe anomalous diffusion on fractals, and physical objects of fractional dimensions, such as some amorphous semiconductors or highly porous materials, random walk deductions, etc.

[1]Department of Mathematics, Alzahra University, Iran
[2]Department of Mathematics, Shahid Beheshti University, Iran

Corresponding author:
Yadollah Ordokhani, Department of Mathematics, Faculty of Mathematical Sciences, Alzahra University, Vanak Street, Tehran 1993893973, Iran.

One of the problems that we face in modelling natural phenomena are fractional optimal control problems and delay fractional optimal control problems. Fractional optimal control problems are considered due to the great application of these problems in engineering, physics, economics, power systems, transportation, biology, electronics, chemistry, and so on. We can see some applications of fractional optimization control problems in Bohannan (2008) and Suárez et al. (2008).

The importance of optimal control problems has led to the development of various analytical and numerical methods to solve these problems. Several efficient numerical methods for solving fractional optimal control problems such as Bernoulli polynomials (Keshavarz et al., 2015), Boubaker polynomials (Rabiei et al., 2017b), multiwavelets (Lotfi, 2011; Yousefi et al., 2011), the Ritz-variational method (Lotfi and Yousefi, 2017), the modified Adomian decomposition method (Alizadeh and Effati, 2017), the hybrid meshless method (Darehmiraki et al., 2018), the finite element method (Zhou and Gong, 2016), methods based on eigenfunctions Özdemir et al., 2009), etc. have been developed.

Another category of fractional problems is the *fractional delay control problem*. Apart from diffusion on fractals, other applications of this kind of problem happen in fields such as fluid flow, viscoelasticity, control theory of dynamical systems, diffusive transport akin to diffusion, electrical networks, probability and statistics, electrochemistry of corrosion, optics and signal processing, rheology, and so on (Bahaa, 2017).

The existence and uniqueness of the solutions of fractional delay differential equations are considered in many studies such as the one by Liao and Ye (2009). However, the delay fractional optimal control problem is a subject that still needs to be investigated, as proved by the poor literature on the topic (Bahaa, 2017).

For delay systems, the optimal control problem is one of the most challenging mathematical problems in control theory. In fact, the presence of delay makes the analysis and design of control much more complicated. Except for some special cases, this is quite complicated or impossible. Therefore, numerical methods are used in most cases (Marzban, 2016; Sweilam and AL-Mekhlafi, 2017). For instance, in Wang et al. (2011), the authors obtained some sufficient conditions for the existence, uniqueness and continuous dependence of mild solutions of fractional finite time-delay evolution systems and optimal controls in infinite-dimensional spaces, and they presented the existence of optimal pairs of fractional time-delay evolution systems. In recent years, various types of delay systems have been considered by many researchers as their optimal control and stability analyses (Barati, 2012; Briat, 2015; Mohan and Kar, 2010).

In many studies, the goal is to obtain the necessary conditions for optimizing and solving delay optimal control problems, such as in Kharatishvili (1961) and Marzban and Hoseini (2015). Applying the Pontryagin maximum principle to the results of the delay optimal control problem in a system of two-point boundary value problems with both delayed and advanced terms, there are no analytic solutions except for some specific cases (Marzban, 2016).

Also, several methods have been used to solve delay fractional optimal control problems, such as methods based on Bernoulli wavelets (Rahimkhani, 2016), fractional-order Boubaker functions (Rabiei et al., 2017a), the Legendre operational technique (Bhrawy, 2016), Bernstein polynomials (Safaie and Farahi, 2014, 2016; Safaie et al., 2014), hybrids of block-pulse functions and Bernoulli polynomials (Haddadi et al., 2012), Legendre multiwavelets (Khellat, 2009), the method in Bahaa (2017), the method in Sweilam and ALMekhlafi (2017), the method of moments (Dehghan and Keyanpour, 2017), the Haar wavelet collocation method (Borzabadi and Asadi, 2013), the method in Göllmann et al. (2009), the method in Koshkouei et al. (2012), hybrids of block-pulse functions and Bernstein polynomials (Dadkhah et al., 2018), and so on.

By considering the zeros of orthogonal polynomials (such as Legendre polynomials, Chebyshev polynomials, etc.) as the nodes of Lagrange polynomials, orthogonal Lagrange polynomials are constructed (Szegö, 1967). Thus we present *fractional-order Lagrange polynomials* (FLPs), the fractional derivative operational matrix of FLPs, and the delay operational matrix of FLPs generally, without considering the nodes of Lagrange polynomials.

As a result, by choosing the different nodes of Lagrange polynomials, we have orthogonal and non-orthogonal fractional-order Lagrange functions. This is one of the important advantages of FLPs. Another advantage of FLPs is the existence of the fractional-parameter, α. In the examples presented, we can see that the effect of this parameter is to solve fractional delay optimal control problems.

This paper is organized as follows. In 'Preliminaries and notations', we describe some necessary definitions and mathematical preliminaries required for our subsequent development. 'Fractional-order Lagrange polynomials' is devoted to recalling fractional Lagrange polynomials. In the next section, we propose a new operational matrix of the Caputo fractional derivative for FLPs. Also, we derive the delay operational matrix in this section. In 'Problem statement', the delay and fractional derivative operational matrices, together with the collocation method at the Lagrange points, are used to solve delay fractional optimal control problems. The convergence of the proposed method and the approximation of FLPs is proposed in 'Convergence analysis'. Also in this section, the upper bound of the error for the operational matrix of fractional derivatives is derived. In 'Illustrative test problems', we report our numerical results by considering some examples. Moreover, we apply the presented method to solving one of the models of chemotherapy in breast cancer. The final section contains some concluding remarks.

Preliminaries and notations

In this section, we give some of the basic definitions and properties of fractional calculus theory that are used in this paper.

Definition 1. Let $f : [a, b] \rightarrow R$ be a function, $\nu > 0$ a real number and $m = \lceil \nu \rceil$, where $\lceil \nu \rceil$ denotes the smallest integer greater than or equal to α, the Riemann–Liouville integral of fractional order is defined as (Sabermahani et al., 2018: p. 5, equations (1)–(4))

$$I^{\nu}f(x) = \begin{cases} \frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}f(t)\mathrm{d}t = \frac{1}{\Gamma(\nu)}x^{\nu-1} * f(x) & \nu > 0, \\ f(x) & \nu = 0 \end{cases} \quad (1)$$

where $x^{\nu-1} * f(x)$ is the convolution product of $x^{\nu-1}$ and $f(x)$.

For the Riemann–Liouville fractional integrals, we have

$$I^{\nu}x^{m} = \frac{\Gamma(m+1)}{\Gamma(m+1+\nu)}x^{\nu+m}, \quad m > -1$$

and

$$(D^{\nu}I^{\nu}f)(x) = f(x),$$
$$(I^{\nu}D^{\nu}f)(x) = f(x) - \sum_{i=0}^{\lceil \nu \rceil - 1}\frac{x^{i}}{i!}f^{(i)}(0)$$

Definition 2. Caputo's fractional derivative of order ν is defined as (Sabermahani et al., 2018: p. 5, equations (5)–(6))

$$D^{\nu}f(x) = \frac{1}{\Gamma(m-\nu)}\int_0^x \frac{f^{(m)}(t)}{(x-t)^{\nu-m+1}}\mathrm{d}t$$

for $m-1 < \nu \leqslant m$, $m \in N$, $x > 0$. For the Caputo derivative, we have

1.

$$D^{\nu}x^{k} = \begin{cases} 0, & \nu \in N_0,\ k < \nu, \\ \frac{\Gamma(k+1)}{\Gamma(k-\nu+1)}x^{k-\nu} & otherwise \end{cases} \quad (2)$$

2. $D^{\nu}\lambda = 0$ where, λ is constant.
3. The Caputo fractional derivative is a linear operation, namely

$$D^{\nu}(\lambda f(x) + \mu y(x)) = \lambda D^{\nu}f(x) + \mu D^{\nu}y(x)$$

where λ and μ are constants.

Definition 3. (Generalized Taylor's formula) (Odibat and Shawagfeh, 2007: p. 4, equation (3.12)). Suppose that $D^{k\alpha}f(x) \in C(0,1]$ for $k = 0, 1, \ldots, n$. Then, we have

$$f(x) = \sum_{k=0}^{n-1}\frac{x^{k\alpha}}{\Gamma(k\alpha+1)}D^{k\alpha}f(0^{+}) + \frac{x^{n\alpha}}{\Gamma(n\alpha+1)}D^{n\alpha}f(\xi)$$

with $0 < \xi \leqslant x$, $\forall x \in (0,\ 1]$. Also, one has

$$|f(x) - \sum_{k=0}^{n-1}\frac{x^{k\alpha}}{\Gamma(k\alpha+1)}D^{k\alpha}f(0^{+})| \leqslant M_{\alpha}\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}$$

where $M_{\alpha} \geqslant \sup_{\xi \in (0,\ 1]}|D^{n\alpha}f(\xi)|$. Also, the generalized Taylor formula reduces to the classical Taylor formula, when $\alpha = 1$.

Fractional-order Lagrange polynomials

In this section, we recall the definition of fractional-order Lagrange functions and their properties.

Lagrange polynomials

Suppose that the set of nodes of Lagrange polynomials is given by $x_i \in [0,1]$, $i = 0, 1, \ldots, n$. A Lagrange polynomial based on these points can be defined as follows (Stoer and Bulirsch, 2002: p. 39, equations (2.1.1.2) and (2.1.1.3))

$$L_i(x) := \prod_{\substack{j=0 \\ i \neq j}}^{n}\frac{(x-x_j)}{(x_i-x_j)} \quad (3)$$

Moreover, the Lagrange polynomials are proposed on the set of nodes $x_i \in [0,\ 1]$ as follows (Sabermahani et al., 2018: p. 7, equations (14)–(16))

$$L_i(x) = \sum_{s=0}^{n}\beta_{is}x^{n-s}, \quad i = 0,\ 1,\ \ldots,\ n \quad (4)$$

where $\beta_{i0} = \frac{1}{\prod_{\substack{j=0 \\ i\neq j}}^{n}(x_i-x_j)}$ and

$$\beta_{is} = \frac{(-1)^{s}}{\prod_{\substack{j=0 \\ i\neq j}}^{n}(x_i-x_j)}\sum_{k_s=k_{s-1}+1}^{n}\cdots\sum_{k_1=0}^{n-s+1}\prod_{r=1}^{s}x_{k_r},$$

$$s = 1, 2, \ldots, n, \quad i \neq k_1 \neq \ldots \neq k_s$$

Fractional-order Lagrange functions

We recall the new set of fractional functions constructed by changing x to x^{α}, $(0 < \alpha \leqslant 1)$. These functions are called FLPs. Using equation (4), the analytic form of FLPs $L_i^{\alpha}(x)$ is as follows (Sabermahani et al., 2018: p. 7, equations (17)–(19))

$$L_i^{\alpha}(x) = \sum_{s=0}^{n}\beta_{is}x^{\alpha(n-s)}, \quad i = 0,\ 1,\ 2,\ \ldots,\ n \quad (5)$$

where $\beta_{i0} = \frac{1}{\prod_{\substack{j=0 \\ i\neq j}}^{n}(x_i-x_j)}$ and

$$\beta_{is} = \frac{(-1)^{s}}{\prod_{\substack{j=0 \\ i\neq j}}^{n}(x_i-x_j)}\sum_{k_s=k_{s-1}+1}^{n}\cdots\sum_{k_1=0}^{n-s+1}\prod_{r=1}^{s}x_{k_r},$$

$$s = 1,\ 2,\ \ldots,\ n, \quad i \neq k_1 \neq \ldots \neq k_s$$

FLPs are obtained with arbitrary nodal points. Thus, we have various choices of Lagrange polynomial nodes to construct fractional-order Lagrange functions.

Function approximation

A function f defined over $[0,1]$ can be expressed in terms of FLPs as

$$f(x) \simeq \sum_{i=0}^{n} c_i L_i^\alpha(x) = C^T L^\alpha(x) \tag{6}$$

where T indicates transposition, and C and $L^\alpha(x)$ are $1 \times (n+1)$ vectors given by

$$C = [c_0, c_1, \ldots, c_n]^T, \qquad L^\alpha(x) = [L_0^\alpha(x), L_1^\alpha(x), \ldots, L_n^\alpha(x)]^T \tag{7}$$

By using equation (6), we have

$$f_j = \langle f, L_j^\alpha \rangle = \left\langle \sum_{i=0}^{n} c_i L_i^\alpha(x), L_j^\alpha \right\rangle = \sum_{i=0}^{n} c_i d_{ij}, \quad j = 0, 1, \ldots, n$$

Suppose that

$$F = [f_0, f_1, \ldots, f_n]^T, \quad D = [d_{ij}]$$

so we have

$$F^T = C^T D$$

$$D = \langle L^\alpha(x), L^\alpha(x) \rangle = \int_0^1 L^\alpha(x)(L^\alpha(x))^T x^{\alpha-1} dx$$

then, we obtain

$$C = D^{-1} \langle f, L^\alpha \rangle$$

Operational matrices of fractional derivative and delay

In this section, we present FLP operational matrices of fractional derivative and delay. We obtain these operational matrices generally, without regard to the nodes of x_i, $i = 0, 1, \ldots, n$.

The fractional derivative operational matrix of FLPs

The fractional derivative of $L^\alpha(x)$ can be approximated as

$$D^\nu L^\alpha(x) \simeq D^{(\nu, \alpha)} L^\alpha(x) \tag{8}$$

$D^{(\nu, \alpha)}$ is called the *fractional derivative operational matrix* of FLPs.

Using equation (5), we obtain

$$D^\nu L_i^\alpha(x) = D^\nu\left(\sum_{s=0}^{n} \beta_{is} x^{\alpha(n-s)}\right) = \sum_{s=0}^{n} \beta_{is} D^\nu x^{\alpha(n-s)}$$

Using equation (2), we have

$$D^\nu x^{\alpha(n-s)} = \begin{cases} 0, & \alpha(n-s) < \nu, \\ \dfrac{\Gamma(\alpha(n-s)+1)}{\Gamma(\alpha(n-s)-\nu+1)} x^{\alpha(n-s)-\nu}, & \alpha(n-s) \geqslant \nu \end{cases}$$

Therefore, we achieve

$$D^\nu L_i^\alpha(x) = \sum_{s=0}^{n} w_{i,s}^\alpha x^{\alpha(n-s)-\nu}$$

where

$$w_{i,s}^\alpha = \begin{cases} 0, & \alpha(n-s) < \nu, \\ \dfrac{\Gamma(\alpha(n-s)+1)}{\Gamma(\alpha(n-s)-\nu+1)} \beta_{is}, & \alpha(n-s) \geqslant \nu \end{cases}$$

We can expand $x^{\alpha(n-s)-\nu}$ in terms of FLPs as

$$x^{\alpha(n-s)-\nu} = \sum_{j=0}^{n} c_{s,j} L_j^\alpha(x) \tag{9}$$

and

$$c_{s,j} = D^{-1} \langle x^{\alpha(n-s)-\nu}, L_j^\alpha(x) \rangle$$

Then, we get

$$\begin{aligned} D^\nu L_i^\alpha(x) &\simeq \sum_{s=0}^{n} w_{i,s}^\alpha \sum_{j=0}^{n} c_{s,j} L_j^\alpha(x) \\ &= \sum_{s=0}^{n} \left(\sum_{j=0}^{n} w_{i,s}^\alpha c_{s,j} \right) L_j^\alpha(x) = \sum_{s=0}^{n} \theta_{i,j,s}^\alpha L_j^\alpha(x) \end{aligned} \tag{10}$$

where $\theta_{i,j,s}^\alpha = \sum_{j=0}^{n} w_{i,s}^\alpha c_{s,j}$. We obtain

$$D^\nu L_i^\alpha(x) \simeq \left[\sum_{s=0}^{n} \theta_{i,0,s}^\alpha, \sum_{s=0}^{n} \theta_{i,1,s}^\alpha, \ldots, \sum_{s=0}^{n} \theta_{i,n,s}^\alpha \right] L^\alpha(x),$$
$$i = 0, 1, \ldots, n$$

Hence, we have

$$D^{(\nu,\alpha)} = \begin{bmatrix} \sum_{s=0}^{n} \theta_{0,0,s}^\alpha & \sum_{s=0}^{n} \theta_{0,1,s}^\alpha & \cdots & \sum_{s=0}^{n} \theta_{0,n,s}^\alpha \\ \sum_{s=0}^{n} \theta_{1,0,s}^\alpha & \sum_{s=0}^{n} \theta_{1,1,s}^\alpha & \cdots & \sum_{s=0}^{n} \theta_{1,n,s}^\alpha \\ \vdots & \ddots & & \vdots \\ \sum_{s=0}^{n} \theta_{n,0,s}^\alpha & \sum_{s=0}^{n} \theta_{n,1,s}^\alpha & \cdots & \sum_{s=0}^{n} \theta_{n,n,s}^\alpha \end{bmatrix}$$

Delay operational matrix of FLPs

Now, we derive a general delay operational matrix based on FLPs. Using equation (4), the Lagrange polynomials vector can be considered as

$$L(x) = \Lambda T_n(x) \tag{11}$$

where

$$T_n(x) = [1, x, x^2, \ldots, x^n]^T, \quad L(x) = [L_0(x), L_1(x), \ldots, L_n(x)]^T$$

and $\Lambda = (\gamma_{i,j})_{i,j=0}^{n}$ is matrix of order $(n+1) \times (n+1)$, where $\gamma_{i,j} = \beta_{i,n-j}$.

Also, for Taylor polynomials, we have (Safaie2: p. 5, equation (8))

$$T_n(x-\tau)=\theta(\tau)T_n(x)$$

where $\theta(\tau)$ is the following matrix

$$\theta(\tau)=\begin{bmatrix} 1 & 0 & & \cdots & 0 \\ -\tau & 1 & & \cdots & 0 \\ (-\tau)^2 & -2\tau & & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ (-\tau)^n & \begin{pmatrix} n \\ n-1 \end{pmatrix}(-\tau)^{n-1} & & \cdots & 1 \end{bmatrix}$$

Using equation (11), we obtain

$$L(x-\tau)=\Lambda\theta(\tau)T_n(x)=\Lambda\theta(\tau)\Lambda^{-1}L(x)$$

Moreover, we can write

$$L(x^\alpha-\tau)=\Lambda\theta(\tau)\Lambda^{-1}L(x^\alpha)$$

and

$$L^\alpha(x-\tau)=\Lambda\theta(\tau)\Lambda^{-1}L^\alpha(x) \tag{12}$$

Now, we consider $M_\tau=\Lambda\theta(\tau)\Lambda^{-1}$, where M_τ is the delay operational matrix of FLPs.

Problem statement

In this section, we employ the operational matrices of delay and the fractional derivative, together with the collocation method to solve the following optimization problem.

Consider the following delay fractional system

$$\begin{aligned} D^\nu X(x) &= A(x)X(x)+B(x)X(x-\tau) \\ &\quad + E(x)U(x)+F(x)U(x-\tau^*)+\delta g(X,U) \end{aligned} \tag{13}$$

with the initial conditions

$$\begin{aligned} X(x)&=\Phi_1(x), \quad -\tau\leqslant x\leqslant 0, \\ U(x)&=\Phi_2(x), \quad -\tau^*\leqslant x\leqslant 0 \end{aligned} \tag{14}$$

where, $0<\nu\leqslant 1, 0\leqslant x\leqslant 1$, δ is constant, $X(x)\in R^l$, $U(x)\in R^q$, $(l\geqslant q)$, $A(x)$, $B(x)$, $E(x)$, and $F(x)$ are continuous matrix functions, and $\Phi_1(x)$ and $\Phi_2(x)$ are known functions defined on the intervals $[-\tau,\ 0]$ and $[-\tau^*,\ 0]$, respectively.

We find the optimal control $U(x)$ and the state trajectory $X(x)$, $0\leqslant x\leqslant 1$ that satisfy equation (13), while extremizing the quadratic performance index (Rahimkhani, 2016: p. 6, equation (26))

$$J=\frac{1}{2}\int_0^1[X^{\mathrm{T}}(x)Q(x)X(x)+U^{\mathrm{T}}(x)R(x)U(x)]\mathrm{d}x \tag{15}$$

where $Q(x)$ a symmetric positive-semidefinite matrix and $R(x)$ a symmetric positive-definite matrix with proper dimensions.

Assuming the existence and uniqueness of the solution of this problem, we solve the system dynamics equation (13) and the performance index equation (15) via FLPs.

We have different choices for $x_i,\ i=0,1,\dots,n$. For example, By considering the zeros of shifted Legendre polynomials as x_i, we have a set of orthogonal polynomials.

In this paper, we consider $x_i=\frac{i}{n},\ i=0,\ 1,\ \dots,\ n$, where these nodes are in $[0,\ 1]$.

We can approximate $X(x)$, $U(x)$, $Q(x)$, $R(x)$, $A(x)$, $B(x)$, $E(x)$ and $F(x)$ in terms of FLPs as

$$\begin{aligned} &X(x)\simeq X^{\mathrm{T}}L^\alpha(x),\quad U(x)\simeq U^{\mathrm{T}}L^\alpha(x),\quad Q(x)\simeq Q^{\mathrm{T}}L^\alpha(x), \\ &R(x)\simeq R^{\mathrm{T}}L^\alpha(x), \\ &A(x)\simeq A^{\mathrm{T}}L^\alpha(x),\quad B(x)\simeq B^{\mathrm{T}}L^\alpha(x),\quad E(x)\simeq E^{\mathrm{T}}L^\alpha(x), \\ &F(x)\simeq F^{\mathrm{T}}L^\alpha(x) \end{aligned}$$

So, we get

$$D^\nu X(x)\simeq X^{\mathrm{T}}D^{(\nu,\ \alpha)}L^\alpha(x), \tag{16}$$

$$X(x-\tau)\simeq X^{\mathrm{T}}M_\tau L^\alpha(x), \tag{17}$$

$$U(x-\tau)\simeq U^{\mathrm{T}}M_{\tau^*}L^\alpha(x) \tag{18}$$

Now, the initial conditions equation (14) can be rewritten

$$\Phi_1(0)\simeq G_1^{\mathrm{T}}L^\alpha(0),\quad \Phi_2(0)\simeq G_2^{\mathrm{T}}L^\alpha(0) \tag{19}$$

We substitute these approximations in the performance index J and the system dynamics equation (13). For the system dynamics equation (13), we obtain

$$\begin{aligned} X^{\mathrm{T}}D^{(\nu,\ \alpha)}L^\alpha(x) &- A^{\mathrm{T}}L^\alpha(x)(L^\alpha(x))^{\mathrm{T}}X \\ &- B^{\mathrm{T}}L^\alpha(x)(L^\alpha(x))^{\mathrm{T}}M_\tau^{\mathrm{T}}X \\ &- E^{\mathrm{T}}L^\alpha(x)(L^\alpha(x))^{\mathrm{T}}U \\ &- F^{\mathrm{T}}L^\alpha(x)(L^\alpha(x))^{\mathrm{T}}M_{\tau^*}^{\mathrm{T}}U \\ &- \delta g(X^{\mathrm{T}}L^\alpha(x),U^{\mathrm{T}}L^\alpha(x))=0 \end{aligned} \tag{20}$$

By employing the collocation method for equation (20) at $x_i,\ i=0,1,2,\dots,n$, we have a system of algebraic equations as

$$\begin{aligned} \tilde{K}=X^{\mathrm{T}}D^{(\nu,\ \alpha)}L^\alpha(x_i) &- A^{\mathrm{T}}L^\alpha(x_i)(L^\alpha(x_i))^{\mathrm{T}}X \\ &- B^{\mathrm{T}}L^\alpha(x_i)(L^\alpha(x_i))^{\mathrm{T}}M_\tau^{\mathrm{T}}X \\ &- E^{\mathrm{T}}L^\alpha(x_i)(L^\alpha(x_i))^{\mathrm{T}}U \\ &- F^{\mathrm{T}}L^\alpha(x_i)(L^\alpha(x_i))^{\mathrm{T}}M_{\tau^*}^{\mathrm{T}}U \\ &- \delta g(X^{\mathrm{T}}L^\alpha(x_i),U^{\mathrm{T}}L^\alpha(x_i)), \end{aligned}$$

where, $\tilde{K}=0$.

For solving this optimization problem, using equation (19), we obtain (Rad et al., 2014: p. 6, equation (3.12))

$$\begin{aligned} \tilde{J}=J+\lambda_i\tilde{K}&+\lambda_{n+1}(\Phi_1(0)-G_1^{\mathrm{T}}L^\alpha(0)) \\ &+\lambda_{n+2}(\Phi_2(0)-G_2^{\mathrm{T}}L^\alpha(0)),\quad i=0,1,\dots,n \end{aligned}$$

where λ_i is the unknown multiplier's coefficient. The necessary condition for deriving the extremum of $\tilde{J}$ is that the following equations hold

$$\frac{\partial \tilde{J}}{\partial X_i} = 0, \;\; i = 0, 1, \ldots, n, \tag{21}$$

$$\frac{\partial \tilde{J}}{\partial U_i} = 0, \;\; i = 0, 1, \ldots, n, \tag{22}$$

$$\frac{\partial \tilde{J}}{\partial \lambda_j} = 0, \;\; j = 0, 1, \ldots, n+2 \tag{23}$$

We can solve equations (21)–(23) using any standard iterative method.

Convergence analysis

In this section, we investigate the convergence analysis of the proposed method and an approximation of FLPs is proposed. Also, we derive the upper bound of the error for the operational matrix of fractional derivatives.

A function $f \in L^2[0,\ 1]$ can be expanded as

$$f(x) \simeq \sum_{i=0}^{n} c_i L_i^{\alpha}(x) = C^{\mathrm{T}} L^{\alpha}(x) = f_n(x)$$

then, we have error function $\widehat{E}(x)$ as follows

$$\widehat{E}(x) = |f(x) - f_n(x)|, \quad x \in [0,\ 1]$$

Theorem 1. Suppose that $D^{k\alpha} f \in C(0, 1]$, $k = 0, 1, \ldots, n$ and $Y_n^{\alpha} = \{L_0^{\alpha}(x), L_1^{\alpha}(x), \ldots, L_n^{\alpha}(x)\}$. If $f_n(x)$ is the best approximation to $f(x)$ out of Y_n^{α}, then the error bound of the approximate solution $f_n(x)$ by using an FLP series would be obtained as follows

$$\| f - f_n \|_2 \leqslant \frac{M_\alpha}{\Gamma((n+1)\alpha + 1)\sqrt{(2n+2)\alpha + 1}} \tag{24}$$

where $M_\alpha = \sup_{x \in [0,1]} |D^{(n+1)\alpha} f(x)|$.

Proof: We define

$$\tilde{f}(x) = \sum_{i=0}^{n} \frac{x^{k\alpha}}{\Gamma(k\alpha + 1)} D^{k\alpha} f(0^+)$$

From the generalized Taylor's formula in Definition 3, we have

$$|f(x) - \tilde{f}(x)| \leqslant \frac{x^{(n+1)\alpha}}{\Gamma((n+1)\alpha + 1)} \sup_{x \in [0,1]} |D^{(n+1)\alpha} f(x)|$$

Since, $\tilde{f}(x)$ is the best approximation of f out of Y_n^{α}, $\tilde{f}(x) \in Y_n^{\alpha}$ and from the above relation, we have

$$\| f - f_n \|_2^2 \leqslant \| f - \tilde{f} \|_2^2 = \int_0^1 |f(x) - \tilde{f}(x)|^2 \mathrm{d}x$$

$$\leqslant \int_0^1 \frac{x^{(2n+2)\alpha}}{\Gamma((n+1)\alpha + 1)^2} M_\alpha^2 = \frac{M_\alpha^2}{\Gamma((n+1)\alpha + 1)^2}$$

$$\int_0^1 x^{(2n+2)\alpha} \mathrm{d}x = \frac{M_\alpha^2}{\Gamma((n+1)\alpha + 1)^2((2n+2)\alpha + 1)}$$

the theorem is proved by taking the square roots. □

With respect to $\frac{1}{\Gamma((n+1)\alpha + 1)\sqrt{(2n+2)\alpha + 1}}$, we can see that the error approaches zero, quickly, when n increases. Therefore, the recent theorem proves the convergence of approximations of FLPs to $f(x)$.

Upper bound of the error for the operational matrix of fractional derivatives

Now, we find an upper bound for the error of $D^{(\nu,\alpha)}$ and show that by increasing in the number of FLPs, this error tends to zero.

The error vector $E^{(\nu)}$ of the operational matrix $D^{(\nu,\alpha)}$ is given by

$$E^{(\nu)} = D^{\nu} L^{\alpha}(x) - D^{(\nu,\alpha)} L^{\alpha}(x), \quad E^{(\nu)} = \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_n \end{bmatrix}$$

from equation (9) and Theorem 1, we have

$$\| x^{\alpha(n-s)-\nu} - \sum_{j=0}^{n} b_j L_j^{\alpha}(x) \|_2 \leqslant \frac{M_\alpha}{\Gamma((n+1)\alpha + 1)\sqrt{(2n+2)\alpha + 1}} \tag{25}$$

then, by using equations (10) and (25), we get

$$\begin{aligned} \| e_i \|_2 &= \| D^{\nu} L_i^{\alpha}(x) - \sum_{j=0}^{n} \sum_{s=0}^{n} w_{i,s}^{\alpha} c_{s,j} L_j^{\alpha}(x) \| \\ &= \| \sum_{s=0}^{n} \frac{\Gamma(\alpha(n-s)+1)}{\Gamma(\alpha(n-s)-\nu+1)} \beta_{is} x^{\alpha(n-s)-\nu} \\ &\quad - \sum_{j=0}^{n} (\sum_{s=0}^{n} w_{i,s}^{\alpha} c_{s,j}) L_j^{\alpha}(x) \| \\ &\leqslant \sum_{s=0}^{n} w_{i,s}^{\alpha} \| x^{\alpha(n-s)-\nu} - \sum_{j=0}^{n} b_j L_j^{\alpha}(x) \| \\ &\leqslant \sum_{s=0}^{n} w_{is}^{\alpha} \frac{M_\alpha}{\Gamma((n+1)\alpha + 1)\sqrt{(2n+2)\alpha + 1}} \end{aligned} \tag{26}$$

where $b_j = \sum_{s=0}^{n} c_{s,j}$.

Therefore, we can see that by increasing the number of FLPs, the error vector $E^{(\nu)}$ tends to zero.

Theorem 2. Suppose $X^{\mathrm{T}} L^{\alpha}(.)$, and $U^{\mathrm{T}} L^{\alpha}(.)$ are approximations of functions $X(.)$ and $U(.)$ using FLPs of degree n, respectively. Then, $(X,\ U)$ converge to the exact solutions as n tends to infinity.

Proof: Similar to the proof given in the Bernstein polynomial in Safaie et al. (2014) (p. 9, Theorem 4.1), the theorem is proved for FLPs.

Illustrative test problems

In this section, we apply the proposed method to solve the following test examples.

Table 1. Comparison of the estimated value of J and CPU time for $\nu = 1$ with the other methods for Example 1.

	Haddadi et al. (2012) $N = 3, M = 9$	Wang (2007b) $k = 3, M = 6$	Dadkhah et al. (2018) $N = 3, M = 7$	Rahimkhani (2016) $K = 2, M = 6$	Present method $n = 4$	$n = 6$
J	0.37310517	0.37311241	0.37311163	0.1027	0.0657	0.0292
CPU time	—	—	—	0.935	0.016	0.031

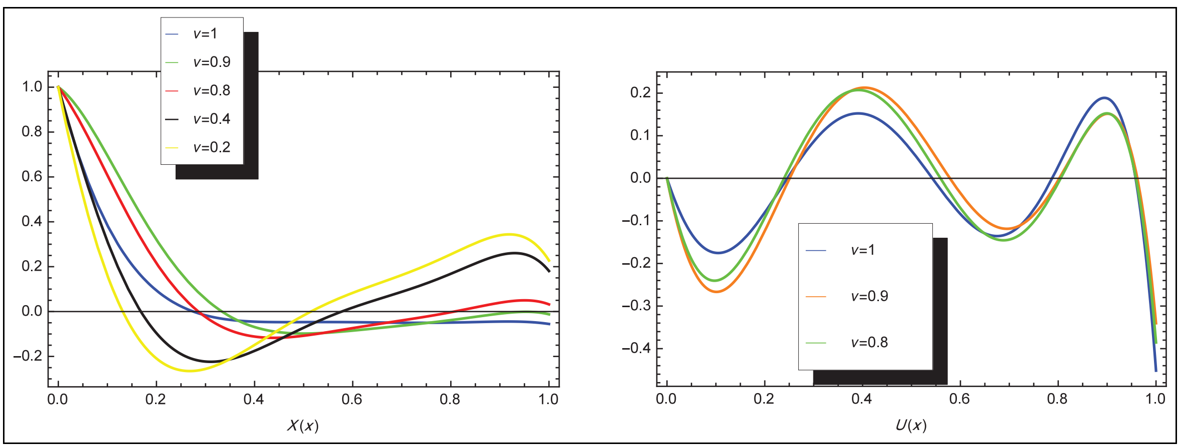


Figure 1. Curves of the approximation of $X(x), U(x)$ for $n = 6, \alpha = 1$ and $\nu = 0.2, 0.4, 0.8, 0.9, 1$, in Example 1.

Example 1. Consider the following *delay fractional optimal control problem* (DFOCP) with different delays in state and control (Dadkhah et al., 2018; Haddadi et al., 2012; Rahimkhani, 2016; Safaie and Farahi, 2014; Wang, 2007b)

$$min \;\; J = \frac{1}{2}\int_0^1 [(X^2(x) + \frac{1}{2}U^2(x)]\mathrm{d}x$$

subject to

$$\begin{aligned} &D^{\nu}X_1(x) = -X(x) + X\left(x - \frac{1}{3}\right) + U(x) - \tfrac{1}{2}U\left(x - \frac{2}{3}\right), \\ &0 \leqslant x \leqslant 1, \quad 0 < \nu \leqslant 1, \\ &X(x) = 1, \quad \frac{-1}{3} \leqslant x \leqslant 0, \qquad U(x) = 0, \quad \frac{-2}{3} \leqslant x \leqslant 0 \end{aligned} \tag{27}$$

Using the present method, we solve this problem with $\tau = \frac{1}{3}, \tau^* = \frac{2}{3}$.

Table 1 displays the approximate values of J obtained by the hybrid of block-pulse functions and Legendre polynomials (Wang, 2007b), the hybrid of block-pulse functions and Bernoulli polynomials (Haddadi et al., 2012), Bernstein polynomials (Safaie and Farahi, 2014), the hybrid of block-pulse functions and Bernstein polynomials (Dadkhah et al., 2018) and Bernoulli wavelets (Rahimkhani, 2016), together with the numerical results of the present method in $\nu = \alpha = 1$ with different values of n. Moreover, in Table 1, CPU times for different values of n are listed.

In this table, we see a good numerical solution derived using the proposed method with a small number of FLPs. Also, Figure 1 shows the approximation of the state variable $X(x)$ and the control variable $U(x)$, respectively, for various values of ν with $n = 6$ and $\alpha = 1$, and this figure demonstrates the convergence of the approximation of $X(x)$ and $U(x)$.

Moreover, the numerical results obtained by the present method for various values of $\alpha = \nu$ and $n = 6$ are shown in Table 2.

Example 2. We consider the following two-dimensional DFOCP (Safaie et al., 2014; Wang, 2007b)

$$min \;\; J = \frac{1}{2}\int_0^1 [(X_1(x) + X_2(x))^2 + U^2(x)]\mathrm{d}t$$

subject to

$$\begin{aligned} &D^{\nu}X_1(x) = X_1(x) + X_2(x - \tfrac{1}{4}) \quad 0 \leqslant x \leqslant 1, \quad 0 < \nu \leqslant 1, \\ &D^{\nu}X_2(x) = -5X_1(x - \tfrac{1}{4}) + X_2(x) - X_2(x - \tfrac{1}{4}) + U(x), \\ &\qquad X_1(x) = X_2(x) = 1, \quad -\tfrac{1}{4} \leqslant x \leqslant 0 \end{aligned} \tag{28}$$

We solve this problem by using the method in 'Problem statement'. Table 3 shows the comparison between the approximations of J obtained by the hybrid of block-pulse functions and Legendre polynomials (Wang, 2007b) with $k = 4$, $M = 4$, and Bernstein polynomials (Safaie et al., 2014) with $m = 6$ together with the present method with $n = 6$, $\nu = 1$ and $\alpha = 1, 0.5$.

For further investigation, we present the CPU time of our method for various values of α in Table 3. This table shows the effectiveness of the method on this problem. Also, we present the numerical results for different values of $\alpha = \nu$ in Table. 4. In this table, we compare the approximate solution with the method based on Bernstein polynomials (Safaie and Farahi, 2016).

Table 2. Estimated value of J with different values of ν for Example 1.

ν	Safaie and Farahi (2014) $m = 6$	Present method $n = 6$
1	0.3956	0.02921
0.99	0.2907	0.02909
0.9	—	0.02909
0.8	—	0.02905
0.4	—	0.20775
0.2	—	0.31851

Table 3. Comparison of the estimated value of J and CPU time for $\nu = 1$, and $\tau = 0.25$ with the other methods for Example 2.

	Wang (2007b)	Safaie et al. (2014)	Present method	
			$\alpha = 1$	$\alpha = 0.5$
J	2.7930	2.6105	2.6105	0.9959
CPU time	—	—	0.031	0.14

Table 4. Estimated value of J for different values of ν, and $\tau = 0.25$ with the other methods for Example 2.

ν	Safaie and Farahi (2016) $m = 6$	Present method $n = 6$
1	1.9493	0.9959
0.9	3.1472	2.7794
0.8	5.8783	4.6060
0.4	—	7.4407
0.2	—	9.9017

Example 3. Consider the following DFOCP (Haddadi et al., 2012; Khellat, 2009; Rahimkhani, 2016; Safaie and Farahi, 2016)

$$min \ J = \frac{1}{2}\int_0^2 [X^2(x) + U^2(x)]\mathrm{d}x$$

subject to

$$\begin{aligned} D^{\nu}X(x) &= xX(x) + X(x-1) + U(x) \quad 0 < \nu \leq 1, \\ X(x) &= 1, \quad -1 \leq x \leq 0 \end{aligned} \tag{29}$$

We rescale the time scale into [0, 1] and by employing the present method, we solve this problem.

In Table 5, a comparison is made between the values of J obtained by Legendre multiwavelets (Khellat, 2009), the method in Rahimkhani (2016), the hybrid of block-pulse functions and Bernoulli polynomials (Haddadi et al., 2012), Bernstein polynomials (Safaie and Farahi, 2016), together with the present method with $n = 4$ for $\nu = 1$ and various values of α. The approximate value obtained using our method for $n = 4$, $\alpha = 1$ is $J = 4.9533$.

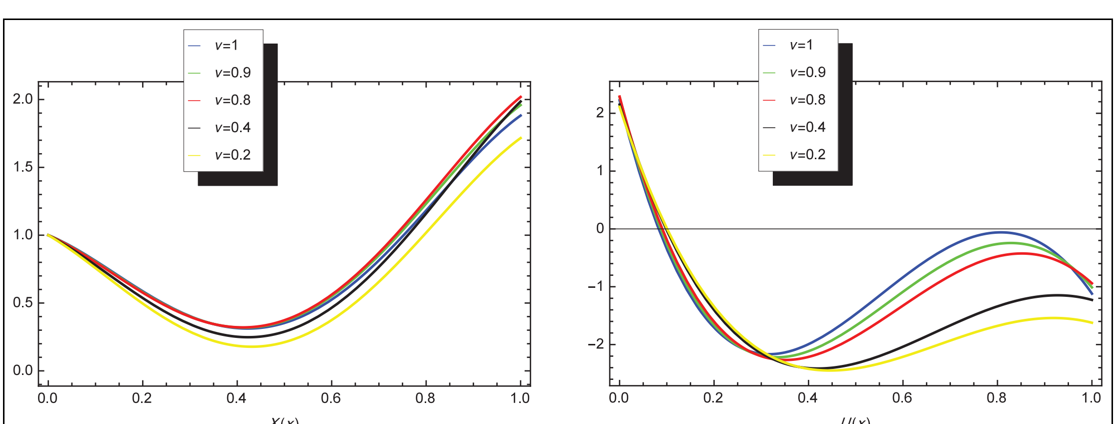
Also, Figure 2 displays the behaviour of the numerical solutions of the state variable $X(x)$ and the control variable $U(x)$ in $\alpha = 1$ and $n = 4$. We can see that as ν approaches 1, the numerical results converge to that of an integer-order differential equation.

Table 6 shows the comparison estimate value of J with different values of ν, $\alpha = 0.25$ obtained by the proposed method and Bernstein polynomials (Safaie and Farahi, 2016).

Tables 5 and 6 and Figure 2 demonstrate the validity and efficacy of our method for this problem. In these tables, we can see that good approximation results are achieved by the present method, with a small number of FLPs.

Table 5. Comparison of the estimated value of J and CPU time with the other methods for Example 3.

	Khellat (2009) $N = 8$	Haddadi et al. (2012) $N = 2, M = 2$	Safaie and Farahi (2016) $m = 6$	Rahimkhani (2016) $k = 2, M = 2$	Present method ($n = 4$) $\alpha = 0.5$	$\alpha = 0.25$
J	5.1713	4.7407	2.7384	2.0481	2.0356	1.7753
CPU time	—	—	—	0.063	0.031	0.016



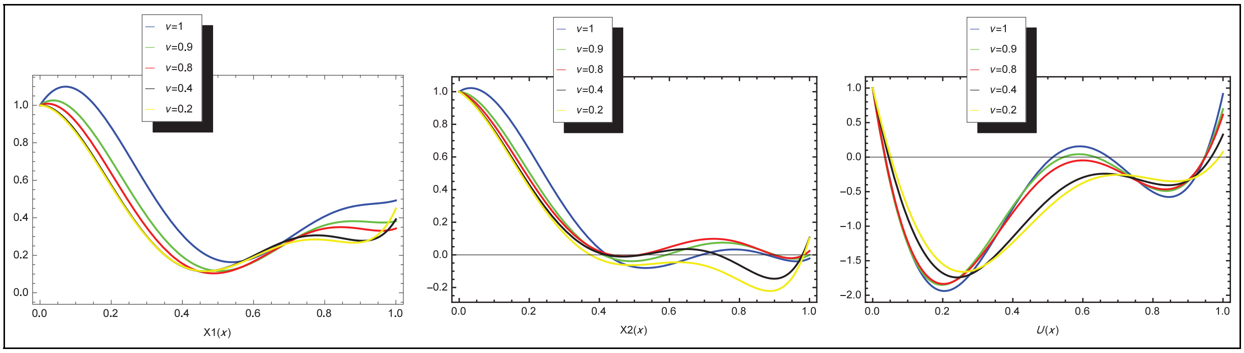


Figure 3. Curves of the approximate solution of $X_1(x), X_2(x), U(x)$, for $n = 6, \alpha = 1$ and different values of ν for Example 4.

Table 6. Estimated value of J with different values of ν for Example 3.

ν	Safaie and Farahi (2016) $m = 6$	Present method $n = 4$
0.9	2.7504	1.9454
0.8	2.8108	2.1763
0.4	—	4.1653
0.2	—	6.6680

Example 4. Consider the following DFOCP (Ghomanjani et al., 2014; Wang, 2007a)

$$D^{\nu}X(x) = \begin{bmatrix} x^2+1 & 1 \\ 0 & 2 \end{bmatrix} X(x-\tau) + \begin{bmatrix} 1 \\ x+1 \end{bmatrix} U(x) + \begin{bmatrix} x+1 \\ x^2+1 \end{bmatrix} U(x-\tau^*), \quad x \in [0,1] \tag{30}$$

in which

$$X(x) = [1,1]^{\mathrm{T}}, \quad -\tau \leqslant x \leqslant 0, \qquad U(x) = 1, \quad -\tau^* \leqslant x \leqslant 0$$

where J is described by

$$J = \frac{1}{2}\int_0^1 \left[X^{\mathrm{T}}(x) \begin{bmatrix} 1 & x \\ x & x^2 \end{bmatrix} X(x) + (x^2+1)U^2(x) \right] \mathrm{d}x$$

We solve this problem with $\tau = \frac{1}{2}, \ \tau^* = \frac{1}{4}$. The exact solution of this problem is unavailable. However, in Figure 3, we show the numerical solution of $X_1(x), X_2(x)$ and $U(x)$ with $n = 6$ and $\alpha = 1$. Figure 3 displays the convergence of the approximate solution, in each case. Also, by applying our method for solving this problem for $n = 4, \nu = 1$, we obtain $J = 1.503157$. However the values of J as derived by methods in Ghomanjani et al. (2014) and Wang (2007a) are $J = 1.536409753$, $J = 1.562240664$, respectively.

Example 5. Consider the following fractional optimal control problem of a nonlinear delay system (Borzabadi and Asadi, 2013; Dadkhah et al., 2018; Göllmann et al., 2009; Koshkouei et al., 2012)

$$min \quad J = 3\int_0^1 [X^2(x) + U^2(x)]\mathrm{d}x$$

subject to

$$D^{\nu}X(x) = 3X(x-\tau)U(x-\tau^*), \quad x \in [0,1] \tag{31}$$

where

$$X(x) = 1, \quad -\tau \leqslant x \leqslant 0, \qquad U(x) = 0, \quad \tau^* \leqslant x \leqslant 0$$

We solve this problem, by employing the present method for $\tau = \frac{1}{3}, \tau^* = \frac{2}{3}, \nu = 1$ with $n = 4$ and different values of α.

Approximate values of J and CPU time obtained by the present method with $n = 4$ are given in Table 7.

In this table we compare the numerical solution with the results derived using the Haar wavelet collocation method with 128 nodes (Borzabadi and Asadi, 2013), the method in Göllmann et al. (2009) by considering 60,000 grid points, the method in Koshkouei et al. (2012), and those derived using a hybrid of block-pulse functions and Bernstein polynomials with $N = 3, M = 9$ (Dadkhah et al., 2018).

Table 8 displays the numerical results achieved by the present method for various values of ν. We can see the effectiveness of this method for solving this problem.

Table 7. Comparison of the estimated value of J and CPU time with the other methods for Example 5.

	Göllmann et al. (2009)	Koshkouei et al. (2012)	Borzabadi and Asadi (2013)	Dadkhah et al. (2018)	Present method $\alpha = 1$	Present method $\alpha = 0.5$
J	3.1082	2.7640	2.761652	2.76140	2.19729	1.48970
CPU time	65.8	7	59.084	—	0.063	0.094

Table 8. Estimated value of J with different values of ν for Example 5.

ν	Present method $n = 4$
0.9	2.3441258
0.8	2.5033788
0.4	3.1714879
0.2	3.9609484

Table 9. Parameters used in the model for Example 6.

a_1	a_4	a_5	a_6	c_1	c_2	c_3
0.98	0.8	0.0001	0.00015	0.9	0.085	0.9
c_4	c_5	c_6	d_1	d_2	d_3	d_4
0.085	0.05	0.00085	0.029	0.11	0.11	0.11

Example 6. Cancer is one of the most dangerous diseases, and causes many deaths each year. The following models the impact of chemotherapy on breast cancer. The model proposed here comes from Newbury (2007), Ramezanpour et al. (2011) and Villasana and Ochoa (2004)

$$J = \frac{1}{2}\int_0^{x_f} [X^2(x) + U^2(x)]\mathrm{d}x$$

subject to following nonlinear delay differential equation

$$\begin{aligned} X'(x) &= A_0X(x) + A_1X(x-\tau) \\ &+ g(X, U), \quad 0 \leqslant x \leqslant x_f, \\ X(x) &= \Phi(x), \quad -\tau \leqslant x \leqslant 0 \end{aligned} \tag{32}$$

where

$$X(x) = [T_Q(x)\ T_I(x)\ T_M(x)\ I(x)]^T,$$

$$U(x) = [u_1(x)\ u_2(x)\ u_3(x)]^T,$$

$$\Phi(x) = [0.01\ 0.015\ 0.01\ 0.01]^T,$$

$$A_0 = \begin{bmatrix} -a_6 - d_4 & 0 & 0 & 0 \\ a_6 & -d_2 & 2a_4 & 0 \\ 0 & 0 & -d_3 - a_4 & 0 \\ 0 & 0 & 0 & -d_1 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a_5 & -a_5 - a_1 & a_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$g(X, U) = \begin{bmatrix} -c_5I(x)T_Q(x) - u_1(x)T_Q(x) \\ -c_5T_I(x)I(x) \\ -c_3T_M(x)I(x) - T_M(x)u_2(x) \\ k + \frac{\rho I(x)(T_Q(x) + T_I(x) + T_M(x))^n}{\alpha + (T_Q(x) + T_I(x) + T_M(x))^n} - c_2I(x)T_I(x) - c_4T_M(x)I(x) - c_6T_Q(x)I(x) - I(x)u_3(x) \end{bmatrix}$$

This model divides the population of tumour cells into interphase cells, mitosis cells, and quiescent cells, which are represented by $T_I(x)$, $T_M(x)$ and $T_Q(x)$, respectively. The term $I(x)$ represents the population of immune cells that are the cytotoxic T-cells. Also $u_1(x)$, $u_2(x)$, $u_3(x)$ are the effects of the chemotherapy drug concentration in the tissue or blood, which are proportional to the dose of drug given to the patient by oral, injection, or in future technology by some kind of portable pump or strap, which can supply drug continuously to the blood circulation.

All parameter values are in fractional amounts per day. The parameter τ is the resident time of cells, which is considered to be 0.09. The constants a_1 and a_4 represent the fraction of cells that change from interphase to mitosis, and from mitosis to interphase, respectively. The constants d_1, d_2 and d_3 represent fractions of apoptosis (natural cell death). The constants c_i represent the loss of cells due to an encounter with other cells. The constants a_5 and a_6 are the transition rates of the proliferating cells to the quiescent cells, and the quiescent cells to the proliferating cells, respectively. The constant d_4 represents the natural death rate of the quiescent tumour cells. Let $n = 2$ and $x_f = 1$. Table 9 summarizes all parameter values in this model.

We apply the present method for solving this problem when $n = 6$, $\alpha = 1, 0.9$. The functional value is $J = 6.3745 \times 10^{-4}$, when $\alpha = 1$. For $\alpha = 0.9$, $J = 1.10035 \times 10^{-8}$. In one day, Figures 4 and 5 demonstrate that the population of tumour cells decreases to zero, whereas immune cells remain at the upper level. These figures also show the effects of the chemotherapy agents.

Conclusion

In this study, we have presented an efficient method for solving DFOCPs. We derived a new fractional derivative operational matrix of FLPs. We have also presented the delay operational matrix of these fractional polynomials. These matrices are generally obtained without considering the nodes of Lagrange polynomials. Operational matrices of fractional derivatives and delay, together with the collocation method were used to approximate solutions of these problems. In fact, we solved the problems directly, without solving the fractional Hamiltonian equations. By giving some examples, we have shown the effectiveness and efficiency of the proposed method.

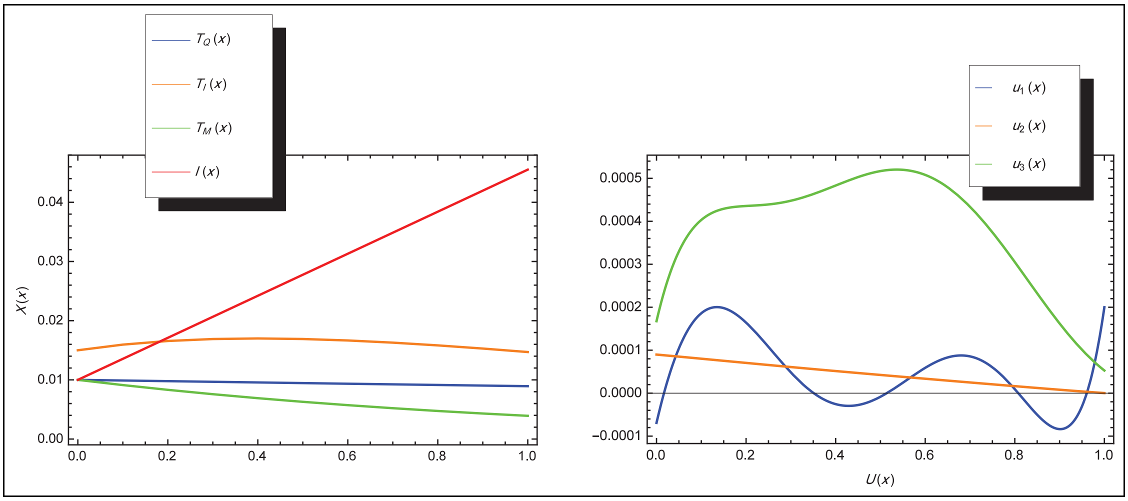


Figure 4. Population of cells during therapy and drug effect on cells with $\alpha = 1$ for Example 6.

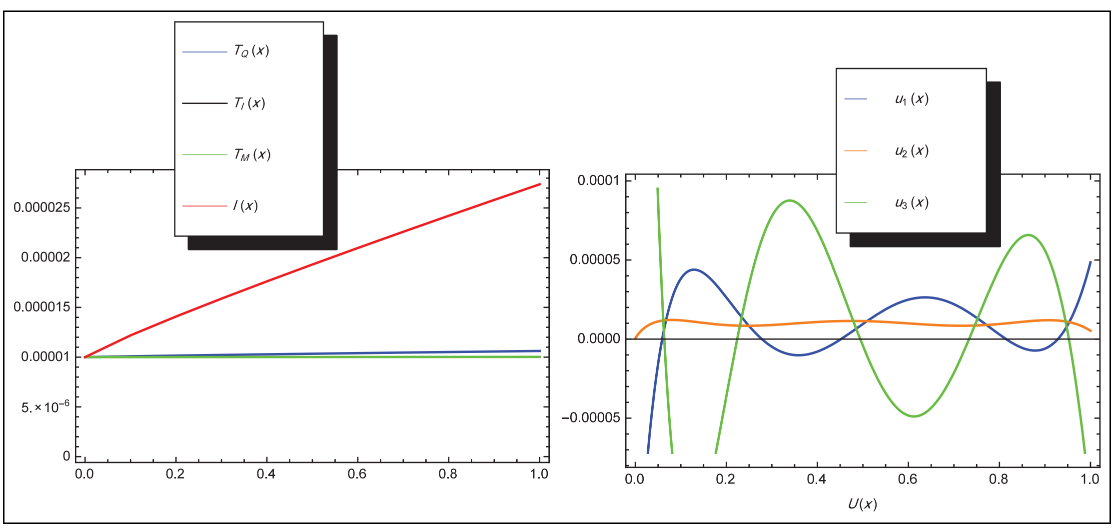


Figure 5. Population of cells during therapy and drug effect on cells with $\alpha = 0.9$ for Example 6.

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Conflict of interest

The authors declare that there is no conflict of interest.

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