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Stability analysis of semilinear stochastic differential equations ☆

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ABSTRACT

This paper is concerned with the global stability of semilinear stochastic differential equations (SDEs) with multiplicative white noise, which is a continuation of our recent work published in SIAM Journal on Control and Optimization, 2018. Under an explicit condition that the Lipschitz constant of nonlinear term is smaller than the top Lyapunov exponent of the linear random dynamical system (RDS), we prove that the zero solution is globally stable.

1. Introduction and main results

Stochastic differential equations (SDEs) have been extensively and intensively studied in many branches, in order to explain the comprehensive effects of interior interactions and noise perturbations, see Arnold (1974), Da Prato and Zabczyk (2014), Friedman (1975), Ikeda and Watanabe (1989), Mao (1997), Øksendal (1998), Prévôt and Röckner (2007). One of the key issues in the study of SDEs is to consider the stability of SDEs, including the almost sure stability, the pth moment stability and so on. During the past several decades, there have been many efforts to develop the stability theory of stochastic ordinary differential equations (Arnold, 1974; Arnold and Schmalfuss, 2001; Kha'sminskii, 1980; Kozin, 1972; Kushner, 1967; Mao, 1999, 1994) and stochastic evolution equations (Haussmann, 1978; Ichikawa, 1982, 1984; Leha et al., 1999; Liu, 1997, 2019; Liu and Mao, 1998; Liu and Mandrekar, 1997; Taniguchi, 1995).

In the finite-dimensional case, Kozin (1972) considered the linear stochastic system and provided a solid foundation for later developments. Also, Kushner (1967) developed the Lyapunov function theory to study strong Markov processes and some control problems. In the meantime, Kha'sminskii (1980) completed a comprehensive work on the stochastic stability theory for the solutions of Itô stochastic ordinary differential equations. For the subsequent research, the reader is referred to Arnold (1974), Arnold and Schmalfuss (2001), Mao (1999, 1994) and others. In the infinite-dimensional

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case, the pioneer work was done by Haussmann (1978) for the linear stochastic system and Ichikawa (1982, 1984) for the semilinear stochastic system, which leads to many subsequent studies, such as Chow (1982), Liu and Mandrekar (1997), Leha et al. (1999), Liu (1997, 2019), Liu and Mao (1998) and Taniguchi (1995). The main tool used here is the Lyapunov functional method, where a major difficulty is to construct suitable Lyapunov functions.

Besides the Lyapunov functional method, another important way of investigating the stochastic stability is to consider the Lyapunov exponent, which has been applied in various stochastic systems, such as Arnold (1998), Arnold et al. (1983), Chueshov (2002), (Furstenberg, 1963a,b), Kha'sminskii (1980) and Mohammed and Scheutzow (1996, 1997). Recently, based on this method and the theory of RDSs, we have showed that the stochastic flow generated by SDEs possesses a globally attracting random equilibrium, which produces the globally stable stationary solution, see Jiang and Lv (2018). The main contribution in Jiang and Lv (2018) is that we established a criteria to guarantee the existence and global stability of nontrivial stationary solutions. In addition, following the same argument and Remark 3 in Jiang and Lv (2018), we can also obtain the global stability of the zero solution. However, to see our main results in Jiang and Lv (2018), we need to assume that the nonlinear function in the drift term is bounded and monotone (or anti-monotone), which cannot be weakened.

In this paper, we shall develop the method of Lyapunov exponents and present a program to deal with the stochastic stability of SDEs. The main purpose of this paper is to consider the global stability of semilinear SDEs with multiplicative white noise, without using the boundedness and monotonicity (or anti-monotonicity). To be specific, we will study the following n-dimensional SDEs

$$dx(t) = [Ax(t) + g(x(t))]dt + \sum_{k=1}^{m} \sigma_k x(t) dB_k(t) \tag{1.1}$$

where $B(t) = (B_1(t), \dots, B_m(t))^{\mathrm{T}}$ is an m-dimensional two-sided Brownian motion on the standard Wiener space $(\Omega, \mathscr{F}, \mathbb{P})$. Here, $\mathscr{F}$ is the Borel σ-algebra of $\Omega = C_0(\mathbb{R}, \mathbb{R}^m) = \{\omega(t)$ is continuous, $\omega(0) = 0,\ t \in \mathbb{R}\}$. In addition, $A = (A_{ij})_{n\times n}$ is an $n \times n$-dimensional matrix, $g : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma_k = (\sigma_{ij}^k)_{n\times n}$ are $n \times n$-dimensional matrices for all $k \in \{1, \dots, m\}$.

In what follows, we set the Euclidean norm $|x| := (\sum_{i=1}^n |x_i|^2)^{\frac{1}{2}}$ and $\|A\| := (\sum_{i=1}^n \sum_{j=1}^n |A_{ij}|^2)^{\frac{1}{2}}$, where $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. For convenience, we first need to introduce some notations. Let θ denote the Wiener shift operator defined by $\theta_t\omega(\cdot) = \omega(t + \cdot) - \omega(t)$ for all $t \in \mathbb{R}$, which is an ergodic metric dynamical system. Furthermore, define $\Psi(t, \omega) = (\Psi_{ij}(t, \omega))_{n\times n}$ for all $\omega \in \Omega$ to be the fundamental matrix of the following linear SDEs

$$dx(t) = Ax(t)dt + \sum_{k=1}^{m} \sigma_k x(t) dB_k(t). \tag{1.2}$$

It is easily seen that the stochastic flow (θ, Ψ) is a linear RDS generated by (1.2), see Arnold (1998), Chueshov (2002). To prove our main results, we will make the following assumptions on A, g and σ_k, $k = 1, \dots, m$:

(A1) The top Lyapunov exponent of (θ, Ψ) is a negative real number. That is, there exists a constant $\lambda > 0$ such that

$$\|\Psi(t, \omega)\| := \left(\sum_{i=1}^{n}\sum_{j=1}^{n} |\Psi_{ij}(t, \omega)|^2\right)^{\frac{1}{2}} \le R(\omega)e^{-\lambda t} \tag{1.3}$$

for all $t \ge 0$ and $\omega \in \Omega$. Here, $R \in \mathcal{L}^1(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{R}_+)$ and

$$\|R\|_{\mathcal{L}^1} = \mathbb{E}R = \int_{\Omega} R(\omega)\mathbb{P}(d\omega).$$

(A2) $g(0) = 0$ and g is globally Lipschitz continuous, i.e.,

$$|g(x) - g(y)| \le L|x - y| \tag{1.4}$$

for all $x, y \in \mathbb{R}^n$, where $L > 0$ is the Lipschitz constant satisfying $\frac{L\|R\|_{\mathcal{L}^1}}{\lambda} < 1$.

Motivated by Jiang and Lv (2018), this paper is concerned with the global stability of the zero solution for (1.1). Define $\psi(t, \omega, x) = x(t, \omega, x)$ to be the unique solution of (1.1) with the initial value $x(0) = x \in \mathbb{R}^n$, we can formulate our main results.

Theorem 1.1. *Assume that* (A1) *and* (A2) *hold, it follows that*

$$\lim_{t\to\infty} \psi(t, \omega, x) = 0 \tag{1.5}$$

for all $x \in \mathbb{R}^n$ and $\omega \in \Omega$.

Corollary 1.1. *Assume that* (A1) *and* (A2) *hold, it follows that*

$$(\mathbb{P}) - \lim_{t\to\infty} \psi(t, \omega, x) = 0 \tag{1.6}$$

and

$$(\mathbb{P}) - \lim_{t\to\infty} \psi(t, \theta_{-t}\omega, x) = 0, \tag{1.7}$$

where the symbol $(\mathbb{P}) - \lim$ *stands for the limit in probability.*

Remark 1.1. In Theorem 1.1, we do not assume that the matrix A is cooperative and g is bounded, monotone (or anti-monotone), which cannot be removed in Jiang and Lv (2018). Moreover, the method used in the proof of Theorem 1.1 is completely different from that in Jiang and Lv (2018).

Remark 1.2. By Theorem 1.1, for the global stability of (1.1), we do not need to construct Lyapunov functions and our conditions (A1) and (A2) are explicit.

2. Proofs of Theorem 1.1 and Corollary 1.1

Proof of Theorem 1.1. Since (θ, Ψ) is a linear RDS, using the variation of constants formula (Mao, 1997, Theorem 3.1), it follows easily that for all $t \geq 0$,

$$\begin{aligned}\psi(t, \omega, x) &= \Psi(t, \omega)x + \Psi(t, \omega)\int_0^t \Psi^{-1}(s, \omega) g\big(\psi(s, \omega, x)\big)ds \\ &= \Psi(t, \omega)x + \int_0^t \Psi(t-s, \theta_s\omega) g\big(\psi(s, \omega, x)\big)ds.\end{aligned} \tag{2.1}$$

Note that $|Mx| \leq \|M\| \cdot |x|$ for all $x \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n\times n}$. Therefore, combining (2.1), (A1) and (A2), we have

$$\begin{aligned}|\psi(t, \omega, x)| &\leq |\Psi(t, \omega)x| + \left|\int_0^t \Psi(t-s, \theta_s\omega) g\big(\psi(s, \omega, x)\big)ds\right| \\ &\leq R(\omega)e^{-\lambda t}|x| + L\int_0^t e^{-\lambda(t-s)} R(\theta_s\omega)|\psi(s, \omega, x)|ds \\ &= R(\omega)e^{-\lambda t}|x| + Le^{-\lambda t}\int_0^t R(\theta_s\omega)e^{\lambda s}|\psi(s, \omega, x)|ds\end{aligned} \tag{2.2}$$

for all $t \geq 0$ and $\omega \in \Omega$. That is,

$$e^{\lambda t}|\psi(t, \omega, x)| \leq R(\omega)|x| + L\int_0^t R(\theta_s\omega)e^{\lambda s}|\psi(s, \omega, x)|ds. \tag{2.3}$$

Let $\varphi(t, \omega, x) := e^{\lambda t}|\psi(t, \omega, x)|$, we see at once that

$$\varphi(t, \omega, x) \leq R(\omega)|x| + L\int_0^t R(\theta_s\omega)\varphi(s, \omega, x)ds,$$

which together with the Gronwall inequality implies that

$$\varphi(t, \omega, x) \leq R(\omega)|x| \exp\left(L\int_0^t R(\theta_s\omega)ds\right). \tag{2.4}$$

This yields that

$$|\psi(t, \omega, x)| \leq R(\omega)|x| \exp\left(-\lambda t + L\int_0^t R(\theta_s\omega)ds\right) \tag{2.5}$$

for all $t \geq 0$ and $\omega \in \Omega$. In addition, using (A1) and the property that θ is an ergodic metric dynamical system, it follows from the Birkhoff–Khinchin ergodic theorem (see (Arnold, 1998, Appendix)) that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t R(\theta_s\omega)ds = \mathbb{E}R = \|R\|_{\mathcal{L}^1} \tag{2.6}$$

for all $\omega \in \widetilde{\Omega}$, where $\widetilde{\Omega}$ is a θ-invariant set of full measure. Without loss of generality, we will still use the symbol Ω to denote $\widetilde{\Omega}$, see Chueshov (2002, p. 13). By (2.6) and (A2), given any $0 < \varepsilon < \frac{\lambda}{L} - \|R\|_{\mathcal{L}^1}$ and $\omega \in \Omega$, there exists $T = T(\omega) > 0$ such that for all $t \geq T$, we get

$$\int_0^t R(\theta_s\omega)ds \leq (\|R\|_{\mathcal{L}^1} + \varepsilon)t,$$

which together with (2.5) shows that

$$|\psi(t,\omega,x)| \le R(\omega)|x|\exp\left[(L(\|R\|_{\mathcal{L}^1}+\varepsilon)-\lambda)t\right] \\ = R(\omega)|x|\exp(-\alpha t), \tag{2.7}$$

where $\alpha = \lambda - L(\|R\|_{\mathcal{L}^1}+\varepsilon) > 0$. This gives that for all $\omega \in \Omega$, $|\psi(t,\omega,x)| \to 0$ as $t \to \infty$. The proof is complete. □

Remark 2.1. The method in the proof of Theorem 1.1 can also be used to prove the global stability of nonlinear stochastic functional differential equations, stochastic evolution equations and so on.

Proof of Corollary 1.1. Using Theorem 1.1, it is evident that (1.6) holds, due to the fact that the convergence almost surely implies the convergence in probability. In addition, since the metric dynamical system θ is a measure preserving flow (see Chueshov (2002, p. 10)), we conclude that (1.7) holds. □

3. An example

Example 3.1. In order to show the validity of our assumptions (A1) and (A2), we will present an example to illustrate our result. Let us consider the following three-dimensional stochastic feedback system

$$\begin{cases} dx_1 = [-5x_1 + \frac{1}{10}\sin x_3]dt + \frac{1}{2}x_1 dB_t^1, \\ dx_2 = [x_1 - 6x_2 + \frac{1}{10}\arctan x_1]dt + \frac{1}{3}x_2 dB_t^2, \\ dx_3 = [-x_2 - 7x_3 + \frac{1}{10}\tanh x_2]dt + \frac{1}{4}x_3 dB_t^3. \end{cases} \tag{3.1}$$

Direct computation shows that the fundamental matrix $\Psi(t,\omega)$ of (3.1) is defined by

$$\Psi(t,\omega) = \begin{bmatrix} \Psi_{11}(t,\omega) & 0 & 0 \\ \Psi_{21}(t,\omega) & \Psi_{22}(t,\omega) & 0 \\ \Psi_{31}(t,\omega) & \Psi_{32}(t,\omega) & \Psi_{33}(t,\omega) \end{bmatrix} \tag{3.2}$$

for all $t \ge 0$ and $\omega \in \Omega$, where

$$\Psi_{11}(t,\omega) = e^{(-5-\frac{1}{8})t+\frac{1}{2}B_t^1(\omega)},$$

$$\Psi_{22}(t,\omega) = e^{(-6-\frac{1}{18})t+\frac{1}{3}B_t^2(\omega)},$$

$$\Psi_{33}(t,\omega) = e^{(-7-\frac{1}{32})t+\frac{1}{4}B_t^3(\omega)}$$

and

$$\Psi_{21}(t,\omega) = \int_0^t e^{(-6-\frac{1}{18})(t-s)+\frac{1}{3}\left(B_t^2(\omega)-B_s^2(\omega)\right)}\Psi_{11}(s,\omega)ds,$$

$$\Psi_{31}(t,\omega) = -\int_0^t e^{(-7-\frac{1}{32})(t-s)+\frac{1}{4}\left(B_t^3(\omega)-B_s^3(\omega)\right)}\Psi_{21}(s,\omega)ds,$$

$$\Psi_{32}(t,\omega) = -\int_0^t e^{(-7-\frac{1}{32})(t-s)+\frac{1}{4}\left(B_t^3(\omega)-B_s^3(\omega)\right)}\Psi_{22}(s,\omega)ds.$$

Therefore, it is easily seen that

$$|\Psi_{11}(t,\omega)| \le e^{-3t}T_1(\omega),\ |\Psi_{22}(t,\omega)| \le e^{-2t}T_2(\omega) \text{ and } |\Psi_{33}(t,\omega)| \le e^{-t}T_3(\omega) \tag{3.3}$$

for all $t \ge 0$ and $\omega \in \Omega$, where

$$T_1(\omega) = \sup_{t\ge 0}\exp\left((-2-\frac{1}{8})t + \frac{1}{2}B_t^1(\omega)\right),$$

$$T_2(\omega) = \sup_{t\ge 0}\exp\left((-4-\frac{1}{18})t + \frac{1}{3}B_t^2(\omega)\right)$$

and

$$T_3(\omega) = \sup_{t\ge 0}\exp\left((-6-\frac{1}{32})t + \frac{1}{4}B_t^3(\omega)\right).$$

This implies that

$$\begin{aligned}|\Psi_{21}(t,\omega)| &= \int_0^t e^{(-6-\frac{1}{18})(t-s)+\frac{1}{3}B^2_{t-s}(\theta_s\omega)}|\Psi_{11}(s,\omega)|ds\\ &\le e^{-2t}T_1(\omega)\int_0^t e^{-s}e^{(-4-\frac{1}{18})(t-s)+\frac{1}{3}B^2_{t-s}(\theta_s\omega)}ds\\ &\le e^{-2t}T_1(\omega)\int_0^\infty e^{-s}T_2(\theta_s\omega)ds\\ &= e^{-2t}T_1(\omega)\widetilde{T}_2(\omega),\end{aligned}$$

$$|\Psi_{31}(t,\omega)| \le e^{-t}T_1(\omega)\widetilde{T}_2(\omega)\int_0^\infty e^{-s}T_3(\theta_s\omega)ds = e^{-t}T_1(\omega)\widetilde{T}_2(\omega)\widetilde{T}_3(\omega)$$

and

$$|\Psi_{32}(t,\omega)| \le e^{-t}T_2(\omega)\int_0^\infty e^{-s}T_3(\theta_s\omega)ds = e^{-t}T_2(\omega)\widetilde{T}_3(\omega).$$

To see conditions (A1) and (A2), we set $\lambda = 1$, $L = \frac{1}{10}$ and

$$\begin{aligned}R(\omega) &= \sqrt{6}\Big[T_1(\omega)\bigvee T_2(\omega)\bigvee T_3(\omega)\bigvee T_1(\omega)\widetilde{T}_2(\omega)\\ &\quad\bigvee T_1(\omega)\widetilde{T}_2(\omega)\widetilde{T}_3(\omega)\bigvee T_2(\omega)\widetilde{T}_3(\omega)\Big]\\ &= \sqrt{6}\Big[T_3(\omega)\bigvee T_2(\omega)\widetilde{T}_3(\omega)\bigvee T_1(\omega)\widetilde{T}_2(\omega)\widetilde{T}_3(\omega)\Big],\end{aligned}$$

where the last equality holds due to the fact that for all $\omega \in \Omega$, $\widetilde{T}_i(\omega) \ge 1$, $i = 2, 3$. Combining the fact that an n-dimensional Brownian motion has n independent components and the property of geometric Brownian motion, i.e., $\mathbb{E}\sup_{t\ge 0}\exp\left(-(\mu+\frac{1}{2}\sigma^2)t+\sigma B_t(\omega)\right) = 1+\frac{\sigma^2}{2\mu}$, where $\mu > 0$ and $\sigma \in \mathbb{R}$, see Graversen and Peskir (1998, p. 585) and Peskir (1998, p. 1639), it is immediate that

$$\begin{aligned}\mathbb{E}R &\le \sqrt{6}\left(\mathbb{E}T_3+\mathbb{E}T_2\cdot\mathbb{E}\widetilde{T}_3+\mathbb{E}T_1\cdot\mathbb{E}\widetilde{T}_2\cdot\mathbb{E}\widetilde{T}_3\right)\\ &= \sqrt{6}\left(\mathbb{E}T_3+\mathbb{E}T_2\cdot\mathbb{E}T_3+\mathbb{E}T_1\cdot\mathbb{E}T_2\cdot\mathbb{E}T_3\right) \qquad (\theta \text{ is measure preserving})\\ &= \sqrt{6}\left(\frac{193}{192}+\frac{73}{72}\cdot\frac{193}{192}+\frac{17}{16}\cdot\frac{73}{72}\cdot\frac{193}{192}\right)\\ &< 7.6115.\end{aligned}$$

Thus, we get that

$$\frac{L\|R\|_{\mathcal{L}^1}}{\lambda} \le \frac{1}{10}\times 7.6115 < 1.$$

That is, (A1) and (A2) hold. Using Theorem 1.1, it follows that the zero solution is globally stable.

Remark 3.1. Example 3.1 shows that the choice of λ and R is a key point in the proof of our result. In addition, the estimate of $\frac{\|R\|_{\mathcal{L}^1}}{\lambda}$ in Example 3.1 may be too large. In fact, it is not easy to get the optimal upper bound of $\frac{\|R\|_{\mathcal{L}^1}}{\lambda}$ for high-dimensional stochastic control systems.

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